Recovery of K-Sparse Non-Negative Signals From K DFT Values and Their Conjugates

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Abstract—The goal is to reconstruct a sparse signal from some, but not all, of its Discrete Fourier Transform (DFT) values. If the signal has K non-zero and real values, a unique solution is determined by any K DFT values, their conjugates, and the DC value, if the DFT order is prime. However, no algorithm is known for this unless the K DFT values are at consecutive frequencies (a total of 2K+1 consecutive values), \( \ell_1 \)-norm minimization only works if the frequencies are less than 4K. It does not use the DC value, provided that the DFT order is prime and \( N/2 \) is also known; otherwise, no algorithm is known only for \( |k| \leq K \) (2K+1 consecutive values of \( X_k \)), then \( x_n \) can be reconstructed using any of the well-known array processing techniques such as Prony’s method, MUSIC, or ESPRIT (even if \( N \) is not prime).

C. Previous Results

It is known that a K-sparse real-valued signal can be recovered uniquely from any K N-point DFT values \( X_k \), their conjugates \( X_{N-k} = X_k^* \), and \( X_0 \), provided that \( N \) is prime. In particular, if \( X_k \) is known for \( |k| \leq K \), then \( x_n \) can be reconstructed using any of the well-known techniques such as Prony’s method, MUSIC, or ESPRIT (even if \( N \) is not prime).

The necessity of \( N \) being prime can be seen as follows. Suppose \( N \) is even. Then knowledge of \( X_{2k} \) for \( |k| \leq K \) would only allow reconstruction of the aliased \( x_{n+N/2} \). Hence there is no way to determine whether \( x_n \) or \( x_{n-N/2} \) is the non-zero element. Replacing 2 with any other factor shows that \( N \) must be prime to ensure uniqueness for any choice of \( k_i \).

A common approach to sparse reconstruction is to compute the minimum \( \ell_1 \) norm solution, usually by linear programming. If the DFT frequencies \( k_i \) are randomly chosen, and if enough of them are known, then it has been shown that the minimum \( \ell_1 \) norm solution is in fact \( x_n \). However, in many practical situations we do not have the luxury of choosing the \( k_i \) at random—they are pre-specified. And the number \( M \) (or 2M+1, depending on how conjugates are counted) must be greater than the minimum of \( K \).

Other approaches include thresholded Landweber iteration and orthogonal matching pursuit. These are much faster computationally, but require more of the problem in order to compute \( x_n \). Since our approach is completely different from all of these, we refer the reader to the extensive literature on these methods.

Our approach, in contrast, works for any choice of frequencies \( k_i \) (and \( N-k_i \)). However, it also requires that \( x_n \geq 0 \) and \( N<4K \), so that the system matrix has aspect ratio=2. This seems to be a common choice in the sparse reconstruction literature.
II. NEW ALGORITHM: PRELIMINARIES

A. Non-Negative Indicator Function

Define the polynomial $S(z)$ as

$$S(z) = \prod_{i=1}^{K} (z - e^{j2\pi n_i/N})(z^{-1} - e^{-j2\pi n_i/N}).$$

$S(z)$ is a polynomial of degree $2K$ whose coefficients $S_k$ have Hermitian symmetry $S_{-k} = S_{k}^{*}$, and $S_k = 0$ for $|k| > K$. Evaluating $S(z)$ on the unit circle $z = e^{j\omega}$:

$$S(e^{j\omega}) = \prod_{i=1}^{K} (e^{j\omega} - e^{j2\pi n_i/N})(e^{-j\omega} - e^{-j2\pi n_i/N}).$$

(3)

Setting $\omega = 2\pi n/N$ shows that

$$S(e^{j2\pi n/N}) = \prod_{i=1}^{K} |e^{j2\pi n_i/N} - e^{j2\pi n_i/N}|^2$$

(4)

from which it is apparent that

$$S(e^{j2\pi n/N}) \begin{cases} = 0 & \text{for } n = n_i \\ > 0 & \text{for } n \neq n_i \end{cases}$$

(5)

The inverse N-point DFT $s_n$ of $S_k$ is seen to be

$$s_n = \frac{1}{N} \sum_{k=-K}^{K} S_k e^{j2\pi nk/N} = \frac{1}{N} S(e^{j2\pi n/N})$$

(6)

$(S_k = 0$ for $|k| > K)$. $s_n$ is real since $S_{-k} = S_{k}^{*}$, and

$$s_n \begin{cases} = 0 & \text{for } n = n_i \\ > 0 & \text{for } n \neq n_i \end{cases}$$

(7)

B. Problem I: Given $\{X_k, |k| \leq K\}$

By Parseval’s theorem for the DFT, we have

$$\sum_{k=-K}^{K} X_k^* S_k = N \sum_{n=0}^{N-1} x_n s_n = N \sum_{i=1}^{K} x_{n_i} s_{n_i} = 0$$

(8)

since $s_{n_i} = 0$ by construction. Furthermore, since $x_n$ and $s_n$ are both non-negative, $\sum_{n=0}^{N-1} x_n s_n = 0$ implies $x_{n_i} s_{n_i} = 0$ for all $n_i$ so that any non-negative K-sparse solution $x_n$ must have $s_{n_i} = 0$ at non-zero locations $n_i$ of $x_n$. So $s_n$ is an indicator function for these.

Suppose we are given the $2K+1$ consecutive values $\{X_k, |k| \leq K\}$. Then $s_n$ can be computed from the $\{X_k, |k| \leq K\}$ by solving for $\{S_k, |k| \leq K\}$ the linear system of inequalities

$$\sum_{k=-K}^{K} X_k^* T_k = 0; Ns_n = \sum_{k=-K}^{K} S_k e^{j2\pi nk/N} \geq 0.$$  

(9)

and looking for zero values of $s_n$. The locations $n_i$ of the zero values of $s_n$ indicate possible non-zero values of $x_n$. Once the $n_i$ have been found, $x_{n_i}$ can be computed either by solving a $K \times K$ linear system of equations, or by using a POCS algorithm.

Of course, Problem I can also be solved using Prony’s method, MUSIC, or ESPRIT, and without requiring non-negativity of $x_n$. But now we extend Problem I to allow missing values of $X_k$ in $|k| \leq K$.

C. Problem II: Given $\{X_k, |k| \leq K+2\}$ Missing $X_{k'}$

Now suppose we are given $\{X_k, |k| \leq K+2\}$ but we are missing $X_{k'+1}$ for some $0 \leq k' \leq K$. None of Prony’s method, MUSIC, or ESPRIT can be applied. But the above procedure can be applied by defining

$$T(z) = S(z)(z + a + z^{-1})^2$$

(10)

In terms of coefficients of $S(z)$ and $T(z)$:

$$(z + a + z^{-1})^2 \sum_{k=-K}^{K} S_k z^{k} = \sum_{k=-K}^{K+2} T_k z^k.$$  

(11)

The constant $a$ is chosen so that $T_{\pm k'}=0$. Then

$$T(e^{j\omega}) = S(e^{j\omega})(a + 2\cos(\omega))^2.$$  

(12)

The $T_{\pm k'}$ are computed from $\{X_k, |k| \leq K + 2, k = \pm k'\}$ by solving for $\{T_k, |k| \leq K + 2\}$ the linear system of inequalities

$$\sum_{k=-K}^{K+2} X_k^* T_k = 0; Nt_n = \sum_{k=-K}^{K+2} T_k e^{j2\pi nk/N} \geq 0.$$  

(13)

We again recognize the inverse DFTs

$$Nt_n = Ns_n(a + 2\cos(2\pi n/N))^2 \geq 0.$$  

(14)

t_n is an indicator for locations of non-zero values of $x_n$. The DFT $T_k$ of $t_n$ has the same properties as $S_k$, except that $T_k = 0$ for $|k| > K+2$ and $T_{\pm k'}=0$.

$t_n$ is computed from $\{X_k, |k| \leq K + 2, k = \pm k'\}$ by solving for $\{T_k, |k| \leq K + 2\}$ the linear system of inequalities

$$\sum_{k=-K}^{K+2} X_k^* T_k = 0; Nt_n = \sum_{k=-K}^{K+2} T_k e^{j2\pi nk/N} \geq 0.$$  

(15)

and looking for zero values of $t_n$.

By induction, we can extend this to any number of missing values of $X_k$. Each missing $X_k$ requires two additional values of $X_k$ to compensate for it. This is disappointing, since in theory we should only need $K$ values of $X_k$ (and their conjugates), even if they are non-consecutive. It may seem as though we could use

$$T(z) = S(z)(z - a)(z^{-1} - a^*)$$

(16)

which maintains $T(e^{j\omega}) \geq 0$. Unfortunately, in this
case it is not always possible to choose $a$ so that $T_{\pm k'}=0$. To see why, expand the additional term into

$$(z-a)(z^{-1}-a^*) = -a^*z + (1+|a|^2) - a z^{-1} \quad (16)$$

so that the constant term must be greater than the square magnitudes of the leading and trailing coefficients. But the Hermitian quadratic polynomial that makes $T_{\pm k'}=0$ does not always have this property.

### III. NEW ALGORITHM: MAIN RESULT

#### A. Problem Statement

We now consider the main problem: Let $x_n$ be a K-sparse real-valued non-negative signal having an N-point DFT $X_k$. Given any 2K+1 values of $X_k$

$$\{X_0, X_{\pm k_1}, X_{\pm k_2} \ldots X_{\pm k_K}\}, \quad 0 < k_i < N/2 \quad (17)$$

we can recover $x_n$ by solving a linear system of inequalities, provided $N < 4K$. $N$ must be prime to ensure uniqueness, but this can usually be relaxed.

#### B. Problem Solution

Since we evaluate polynomials at $z = e^{j2\pi n/N}$, we can perform all polynomial calculations mod($z^N-1$).

So we can redefine $T(z)$ as (compare to above)

$$T(z) = S(z) \prod_{k=1}^{[N/4]} (z+a_k+z^{-1})^2 \mod(z^N-1) \quad (18)$$

where $[N/4]$ is the greatest integer not exceeding $N/4$. Then the extra term is a polynomial of degree $4[N/4] \leq N$, and the $[N/4]$ constants $a_k$ can be chosen to make $[N/4]$ values of $T_{\pm k_i}=0$, so that $X_{\pm k_i}$ are not needed to compute the indicator function $t_n$.

We already know that a K-sparse $x_n$ requires K values of $X_{\pm k}$ and $X_0$. So if $N=4K$, we can zero out K=N/4 values of $T_{\pm k}$, leaving K=N/4 nonzero values of $T_{\pm k}$ and $T_0$, just what we need to compute $x_n$.

Using the same argument as before, the indicator function $t_n$ can be computed by solving the linear system of inequalities

$$\sum_{k=-[N/2]}^{[N/2]} X_k^* T_k = 0; \quad N t_n = \sum_{k=-[N/2]}^{[N/2]} T_k e^{j2\pi nk/N} \geq 0,$$

and looking for zero values of $t_n$.

#### C. Solving Linear Systems of Inequalities

Superficially, this looks like a linear programming problem, but it’s not. It is the first stage of a linear programming problem using the simplex method, in which a basic feasible solution is computed.

However, this problem can also be solved using a POCS algorithm, consisting of the following steps:

- Project $t_n$ onto set of non-negative signals $t_n \geq 0$ using two FFTs to relate $T_k$ to $t_n$;
- Set $\{T_k, k \neq \pm k_i\} = 0$. That is, set $T_k = 0$ for those values of $k$ for which $X_k$ is unknown;
- Project non-zero $\{T_k, k = \pm k_i\}$ onto the known hyperplane $\sum_{k=\pm k_i} X_k^* T_k = 0$;
- Continue until $N/4$ values of $t_n$ are smaller than the remaining $3N/4$ values of $t_n$;
- These $t_n$ are possible locations of non-zero $x_n$.

Once the possible locations $n_i$ of the non-zero values of $x_n$ have been found by thresholding $t_n$, another POCS algorithm can be used to solve for these $x_n$:

- Set $z_n = 0$ for $n \neq n_i$ and for $z_n < 0$;
- Compute DFT $Z_k$ of $z_n$ and set $Z_{\pm k_i} = X_{\pm k_i}$;
- Compute the inverse DFT $z_n$ of $Z_k$;
- Continue until convergence.

### IV. REVIEW OF TWO FFTS

We review two FFT algorithms for fast computation of the DFT that are pertinent to this algorithm.

#### A. Rader Prime Factor FFT

The Cooley-Tukey FFT is well-suited for values of $N$ that are highly composite, e.g., powers of two.

$N$ prime is a worst case for it, since $N$ cannot be factored. But the Rader FFT can be used instead.

Let $N$ be prime and $p$ be a primitive root of $N$ (often $p=2$ can be used). Then

$$\{p^i, i = 1 \ldots N-1\} = \{i, i = 1 \ldots N-1\}, \mod(N). \quad (19)$$

That is, the powers of $p \mod(N)$ are a reordering of $1 \ldots N-1$. Then the DFT can be rewritten as

$$X_{p^k} - x(0) = \sum_{\bar{n}=1}^{N-1} x_{\bar{n}} p^\bar{n} e^{-j2\pi p^{\bar{n}+k}} \quad (20)$$

where $p^\bar{n} \equiv \bar{n} \mod(N)$. The DC value $X_0$ is computed separately as $X_0 = \sum_{n=0}^{N-1} x_n$.

This is a cyclic convolution of order $N-1$, so it can be computed using an $(N-1)$-point DFT. This is especially effective if $N$ is one more than a power of two, e.g., $N=257$ or 65537.

#### B. Good-Thomas 2-D to 1-D FFT

The Good-Thomas FFT is a mapping between a 1-D and 2-D DFT using a residue number system to relate the indices. Let $N = N_1 N_2$ where $N_1$ and $N_2$
are relatively prime. Then define

\[ n \equiv n_1 \mod(N_1); \quad n \equiv n_2 \mod(N_2) \]  

(21)

The \( N_1 N_2 \)-point 1-D DFT \( X_k \) of \( x_n \) is related to the \((N_1 \times N_2)\)-point 2-D DFT \( X_{k_1, k_2} \) of \( x_{n_1, n_2} \), where

\[ k \equiv N_1 k_2 + N_2 k_1 \mod(N_1 N_2) \]  

(22)

So a 2-D DFT can be reformulated as a 1-D DFT, and the algorithm of this paper applied to images.

V. NUMERICAL EXAMPLE

We present a very small (\( N=101 \)) example to illustrate the algorithm. The locations of known \( X_k \) and non-zero \( x_n \) are chosen randomly (while preserving conjugate symmetry locations), although randomness is unnecessary. The non-zero values of \( x_n \) are one, to facilitate the results. The total number of known \( X_k \) is 55 (27 values, their conjugates, and the DC value), and the number of non-zero \( x_n \) is 27.

The result of the method of frames (setting unknown \( X_k = 0 \)) is shown in Figure #1. The unfilled circles are the true values of \( x_n \), and the filled circles are the result of the method of frames. It can be seen that a threshold would recover many, but not all, of the locations of non-zero \( x_n \).

The result of our algorithm is shown in Figure #2. Again the unfilled circles are the actual \( x_n \), and the filled circles are the \( t_n \) computed from the known \( X_k \). It can be seen that zero values of the \( t_n \) find all of the locations of non-zero \( x_n \) (and some other possible values). The total number of zero values of \( t_N \) is 39, 12 more than the actual number of non-zero \( x_n \), but fewer than the 55 known values of \( X_k \). Hence a second POCS algorithm can be used to reconstruct the actual values of the non-zero \( x_n \). The results of this algorithm are not shown.

The Matlab program used to generate these results:

```matlab
clear;N=101;rand('seed',0);
K=find(round(rand(1,floor(N/2)))==1);
K=[1 K fliplr(N+2-K)];
rand('seed',0);X=round(rand(1,N)-1/4);
FX=fft(X);Y=zeros(1,N);Y(K)=FX(K);
%GOAL: Reconstruct sparse X from Y.
F=rand(1,N); %Initialization
for I=1:1000; %Iteration
  %Project S onto S>0:
  S=real(ifft(F));S=S.*(S>0);F=fft(S);
  %Set F=0 except where Y is nonzero:
  F=F.*(Y==0); %Project F onto F*Y=0:
  F=F-Y*(F*Y')/(Y*Y');end; figure
stem(X),hold on,stem(S*100,'filled')
```

![Fig. 1. Result of Method of Frames](image1.png)

![Fig. 2. Result of Algorithm: Indicator](image2.png)