Closed-Form Reconstruction of Sparse Signals from Limited Numbers of Irregular Frequencies

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Abstract— **The goal is to reconstruct a sparse signal or image from some, but not all, of its discrete Fourier transform (DFT) values. We derive a condition that ensures the sparse solution can be computed in closed form by thresholding the inverse DFT of the known DFT values, with unknown DFT values set to zero.**

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I. INTRODUCTION

A. Problem Statement

The N-point DFT X_k of the length=N signal x_n is

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N}, k = 0 \ldots N-1. \quad (1)$$

We are given the following facts about the problem:

- x_n=0 unless $n \in \{n_1 \ldots n_K\}$ (the n_i are unknown);
- X_k only known for $k \in \{k_1 \ldots k_M\}$ (k_i are known);
- $|x_{\text{MIN}}| \leq |x_{n_i}| \leq |x_{\text{MAX}}|$ (bounded nonzero x_n).

The goal is to recover the K-sparse signal x_i from its DFT values X_k known at M frequencies k_i. Since M<N, the problem is very underdetermined, so the sparsity of x_n is used to compute a unique solution.

B. Significance of Results

Reconstruction of signals and images from limited frequency data occurs in problems such as:
- Limited-angle tomography in medicine;
- Synthetic-aperture radar (SAR) imaging;
- Magnetic resonance imaging (MRI).

Many signals and images of practical interest have a sparse representation $\hat{x}_{mn}$ in a wavelet basis, where

$$\hat{x}_{mn} = \sum_{i=0}^{N-1} x_i \psi(2^m i - n). \quad (2)$$

- $\psi(n)$ is a wavelet basis function;
- $\hat{x}_{mn}$ is sparse (mostly zero);
- The DFT of $\hat{x}_{mn}$ is $X_k \Psi(\frac{2\pi k}{2^m N})$.

Hence we can recover the sparse $\hat{x}_{mn}$ from the known X_k at various scales m, and compute x_n from $\hat{x}_{mn}$.

II. LEAST-SQUARES RECONSTRUCTION

A. Derivation of Thresholding Algorithm

Define s_n and its DFT S_k as

$$S_k = \begin{cases} S_k & \text{for } k \in \{k_i\} \\ 0 & \text{for } k \notin \{k_i\} \end{cases} \quad (3)$$

where the known values of S_k are to be determined. Then observing X_{k_i} is equivalent to observing Y_k:

$$Y_k = X_k S_k \Leftrightarrow y_n = \sum_{i=1}^{K} x_{n_i} s_{n-n_i} \quad (4)$$

so each x_{n_i} becomes a scaled and translated s_n. Let

$$S = \max_{n \neq 0} |s_n| = \max_{n \neq 0} \frac{1}{M} |\sum_{m=1}^{M} S_{k_m} e^{j2\pi nk_m/N}| \quad (5)$$

and note that by the triangle inequality we have

$$|y_n| \leq \sum_{i=1}^{K} |x_{n_i}| \cdot |s_{n-n_i}| \leq K|x_{\text{MAX}}|S. \quad (6)$$

Then we have the following:

- $\max\{|y_n|, n \notin \{n_i\}\} = |x_{\text{MAX}}|SK$;
- $\min\{|y_n|, n \in \{n_i\}\} = |x_{\text{MIN}}| - |x_{\text{MAX}}|SK$.

Thresholding $|y_n|$ will identify nonzero locations n_i if

$$|x_{\text{MIN}}| - |x_{\text{MAX}}|SK > |x_{\text{MAX}}|SK \quad (7)$$

which can be rewritten as the sparsity condition

$$K < \frac{1}{2S} \frac{|x_{\text{MIN}}|}{|x_{\text{MAX}}|} \quad (8)$$

Once the locations n_i of nonzero x_n have been identified, x_{n_i} can be computed by solving an overdetermined $M \times K$ linear system of equations (1) with $k \in \{k_i\}$ and $n \in \{n_i\}$. Or, a projection on convex sets (POCS) algorithm can be used here, as follows:

- Given $\hat{x}_n^L$ at iteration #L:
- Compute $\hat{X}_k^L$ from $\hat{x}_n^L$ using (1);
- Set $\hat{X}_{k_i}^L = X_{k_i}$=known X_k;
- Compute $\hat{x}_n^{L+1}$ from $\hat{X}_k^L$ using (1);
- Set $\hat{x}_n^{L+1} = 0$ for $n \notin \{n_i\}$. Continue.

III. PERFORMANCE IMPROVEMENT

Clearly we would like s_n to resemble an impulse as much as possible. There are two ways to achieve this:

- Choose $\{k_i\}$ to minimize *coherence* S;
- Choose S_{k_i} to make $s_n \approx \delta_n$.

A. Estimation of Coherence

The coherence S is a figure of merit for the utility of various frequency location configurations. It is well known that choosing k_i randomly (while maintaining conjugate symmetric locations) minimizes S.

Let the presence or absence of a specific frequency k in the set of given frequencies $\{k_i\}$ be a Bernoulli process with probability of success$=\frac{M}{N}$. Then the expected number of known locations k_i is M, and S_k is an independent and independently distributed (iid) random process with $\sigma^2_{S_k} = \frac{M}{N}(1-\frac{M}{N})$.

The inverse DFT from S_k to s_n has a factor $\frac{1}{M}$ instead of the $\frac{1}{\sqrt{N}}$ that would make it unitary. Hence

$$\sigma^2_{s_n} = \left(\frac{\sqrt{N}}{M}\right)^2 \sigma^2_{S_k} = \frac{N}{M^2}\frac{M}{N}\left(1-\frac{M}{N}\right) \quad (9)$$

and a reasonable estimate of the coherence S is

$$S \approx \sqrt{2}\sigma_{s_n} = \sqrt{\frac{2}{M}-\frac{2}{N}}. \quad (10)$$

Of course, this is independent of the choice of x_n.

B. Estimation of Minimum #Observations

Let x_n be Bernoulli with probability of success$=\frac{K}{N}$. Then the expected number of nonzero values is K. Let nonzero values of x_n be ± 1, so

$$x_n = \begin{cases} +1 & \text{probability } \frac{K}{2N} \\ -1 & \text{probability } \frac{K}{2N} \\ 0 & \text{probability } 1-\frac{K}{N} \end{cases} \quad (11)$$

Then $\sigma^2_{x_n} = \frac{K}{N}$ and (5) shows $\sigma^2_{y_n}$=$N\sigma^2_{s_n}\sigma^2_{x_n}$ and so

$$\sigma^2_{y_n} = N\left(\frac{2}{M}-\frac{2}{N}\right)\frac{K}{N} = 2\frac{K}{M}\left(1-\frac{M}{N}\right) \quad (12)$$

assuming everything is zero-mean and uncorrelated.

Thresholding y_n will work if $\sqrt{2}\sigma_{y_n} < \frac{1}{2}$, or

$$\frac{K}{M}\left(1-\frac{M}{N}\right) < \frac{1}{16}. \quad (13)$$

Note the following:

- If $M \approx N/2$ this becomes $K < M/8$;
- This is consistent with reported bounds;
- Note the absence of any $\log(M)$ term:
- That applies to matrices with Gaussian entries.

C. Choice of Nonzero S_k: Approximation Theory

However, we do not always have the luxury of being able to choose the observed frequencies to be random. In particular, it may be physically impossible to measure high frequencies (this is often the case in optics). So now we investigate choosing $S_{k_i} \neq 1$.

First we consider a *minimax criterion*

$$\min_{S_{k_i}} \max_{n\neq 0} |s_n/s_1| = \min_{S_{k_i}} S. \quad (14)$$

The idea is to choose S_{k_i} to minimize the coherence. This is similar to FIR filter design in digital signal processing, except the filter support is no longer contiguous, and the desired response is now impulsive.

We can apply the theory by solving the system

$$\sum_{i=1}^{M} S_{k_i} e^{j2\pi n k_i/N} = N\delta_n - \epsilon(-1)^n \quad (15)$$

which can be rearranged into the linear system

$$\begin{bmatrix} 1 & \cdots & 1 & 1 \\ \cos(2\pi k_1/N) & \cdots & \cos(2\pi k_{M/2}/N) & -1 \\ \cos(2\pi 2k_1/N) & \cdots & \cos(2\pi 2k_{M/2}/N) & 1 \\ \vdots & \cdots & \vdots & \vdots \\ \cos(2\pi\frac{N}{2}\frac{k_1}{N}) & \cdots & \cos(2\pi\frac{N}{2}\frac{k_{M/2}}{N}) & (-1)^{N/2} \end{bmatrix} \times \begin{bmatrix} S_{k_1} \\ S_{k_2} \\ \vdots \\ S_{k_{N/2}} \\ -\epsilon \end{bmatrix} = \begin{bmatrix} N \\ N/2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix}\text{the least} \\ \text{squares}\end{matrix} \text{ solution.} \quad (16)$$

The alternation theorem of approximation theory states that a minimax solution criterion will oscillate between $\pm\epsilon$ for some ϵ. According to the Remez exchange theorm, the points at which the criterion$=\pm\epsilon$ can be found by assuming they are equally spaced, as above, and then iteratively choosing the points to be the maxima and minima of the criterion at the previous iteration. This is the basic idea behind the Parks-McClellan equiripple filter design procedure.

In practice, attempting to approximate $s_n \approx \delta_n$ leads to relatively large coherence. If we alter the definition of coherence to (4 is only illustrative here)

$$S = \max_{|n|>4} |s_n| \ vs.\ S = \max_{n\neq 0} |s_n| \quad (17)$$

this new S can be made much smaller, at the price of a loss of resolution of closely-spaced nonzero x_n.

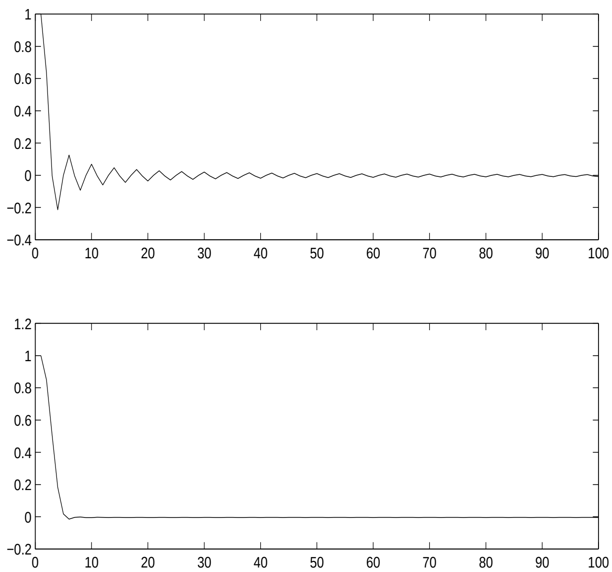
To demonstrate the improvement in coherence, consider the problem where X_k is only known on the lower half-band. Specifically, N=1024, M=512, and X_k is known only for $1 \leq |k| \leq 256$ (DC is excluded).

The coherence with $S_{k_i} = 1$ is the upper plot in Figure #1. Only the first 100 values are plotted. The large sidelobe will produce false nonzero x_n.

The coherence with S_{k_i} computed as above is the lower plot in Figure #1. Again only the first few values are plotted. The sidelobes are much smaller.

The Matlab code used to generate this example:

```
clear;N=1024;
K=[1:N/4];K=[K N-fliplr(K)];
M=length(K);F(N)=0;F(K+1)=1;
S=N/M*real(ifft(F));plot(S)
A=[cos(2*pi*[0:N/2]'*K(1:M/2)/N)];
A=[A -(-1).^[0:N/2]'];%System matrix.
G=A\[N;0.8*N;0.5*N;0.2*N;zeros(N/2-3,1)];
H(K+1)=[G(1:M/2);flipud(G(1:M/2))];
FH=real(fft(H,N));FH=FH/FH(1);plot(FH)
```



D. Choice of Nonzero S_k: Hamming Window

Based on examining the values of H in the above result, we propose to use a *Hamming* (or some other) data window on the range of observed frequencies:

$$S_k = \begin{cases} 0.54 + 0.46\cos(\pi k/L) & \text{for } k \in \{k_i\} \\ 0 & \text{for } k \notin \{k_i\} \end{cases} \quad (18)$$

where $\{k_i\} \subset \{|k| \leq L\}$ and index $N-k$ is now $-k$.

The effect of this change on s_n can be seen in Figure #2. The upper plot is s_n when M=65 out of N=128 known frequencies are all concentrated in $|k| \leq 32$. The sidelobes can create false positives for locations n_i of nonzero x_n. The lower plot shows the effect of a Hamming window. The sidelobes are virtually eliminated. Note that unlike typical filter design problems, *all* sidelobes, not just the ones closest to the main lobe, can interfere with other x_n estimates.

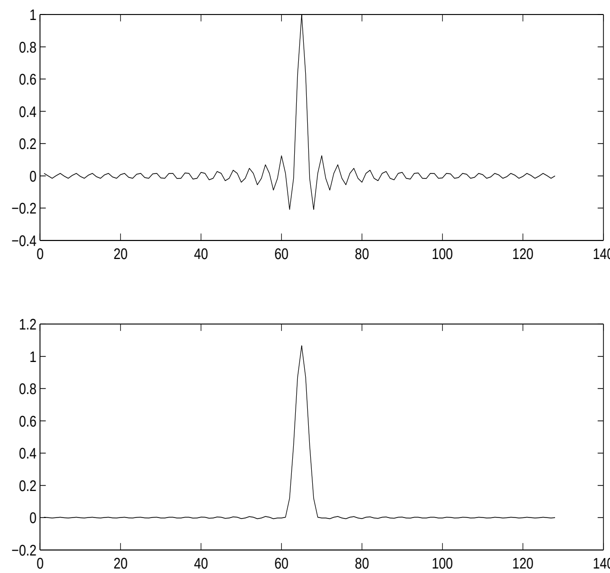
The effect of the Hamming window on sparse reconstruction by thresholding can be seen in Figure #3. Both plots feature K=9 out of N=128 nonzero x_n, equally spaced, with random zero-mean Gaussian values, reconstructed from M=65 lowest frequencies $|k| \leq 32$. This tiny example is used for clarity.

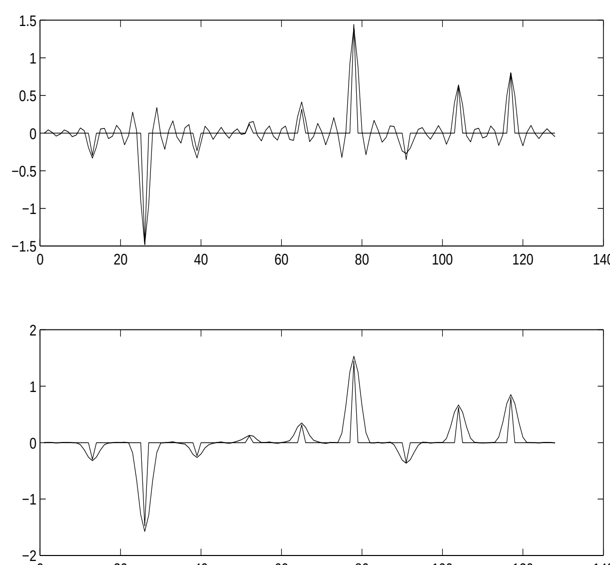
The smaller nonzero x_n are obscured by sidelobes of other nonzero x_n in the upper plot, which uses $S_{k_i} = 1$. But all nonzero x_n are apparent in the lower plot, which uses a Hamming window.

In both cases the broad main lobe limits resolution of two closely-spaced nonzero x_n. Candidate locations n_i can be determined, and x_{n_i} can be reconstructed, with some of them turning out to be zero.

The Matlab code used to generate this example:

```
clear;X(13:13:117)=randn(1,9);X(128)=0;
FX=fft(X,128);H=hamming(65);H=H';
F1=128/65*[ones(1,33) zeros(1,63) ones(1,32)];
F2=256/65*[H(33:65) zeros(1,63) H(1:32)];
Y1=real(ifft(FX.*F1));plot(1:128,X,1:128,Y1)
Y2=real(ifft(FX.*F2));plot(1:128,X,1:128,Y2)
```





IV. Sampling Diversity

A. Problem Statement

A completely different solution to the problem of reconstructing sparse signals from frequency samples is to choose the samples to be two different sets of equally-spaced frequencies with different spacings:

$$\begin{aligned}\{k_i\} &= \{N_1 n_1, 0 \leq n_1 < N/N_1\} \\ &\cup \{N_2 n_2, 0 \leq n_2 < N/N_2\}\end{aligned} \qquad (19)$$

where N_1 and N_2 are relatively prime (no common factors) and both are factors of the length N of x_n. For example, if N=20 we might be given these X_k:

$$\{X_0, X_5, X_{10}, X_{15}\} \cup \{X_4, X_8, X_{12}, X_{16}\}. \qquad (20)$$

B. Solution Procedure

Computing the $\frac{N}{N_1}$ and $\frac{N}{N_2}$-point inverse DFTs

$$x_n^1 = \frac{N_1}{N} \sum_{n_1=0}^{N/N_1-1} X_{N_1 n_1} e^{j\frac{2\pi n n_1}{N/N_1}} \qquad (21)$$

$$x_n^2 = \frac{N_2}{N} \sum_{n_2=0}^{N/N_2-1} X_{N_2 n_2} e^{j\frac{2\pi n n_2}{N/N_2}}$$

gives the two aliased reconstructions

$$x_n^1 = \sum_{i=0}^{N_1-1} x_{n-iN/N_1}; x_n^2 = \sum_{j=0}^{N_2-1} x_{n-jN/N_2} \qquad (22)$$

where the indices are reduced mod(N). Then

$$(x_n^1)(x_n^2) = x_n^2 = 0, \quad n \notin \{n_i\} \qquad (23)$$

(no sign ambiguity: $x_n^1 = x_n^2$ when both are nonzero) if N is large enough that there is no solution to

$$n_1 - iN/N_1 = n_2 - jN/N_2. \qquad (24)$$

C. Numerical Example

A 16-sparse 144×144 image is to be reconstructed from its downsampled ↓(3×3) and ↓(4×4) 2-D DFTs. This is $(\frac{144}{3})^2 + (\frac{144}{4})^2$=3600 observations in $(144)^2$=20736 unknowns. The aliased reconstructions x^1 and x^2 and their product are shown; the latter matches the (not shown) original image exactly.

The Matlab code used to generate this example:

```
clear;rand('seed',0);N=144;N1=4;N2=3;
X=round(rand(N,N)-0.49908);%16-sparse
FX=fft2(X);FX1=FX(1:N1:N,1:N1:N);
FX2=FX(1:N2:N,1:N2:N);%DATA: FX1,FX2.
%GOAL: Reconstruct X from FX1 and FX2.
FY1(1:N1:N,1:N1:N)=FX1;FY1(N,N)=0;
FY2(1:N2:N,1:N2:N)=FX2;FY2(N,N)=0;
X1=N1*N1*real(ifft2(FY1));%Aliased X.
X2=N2*N2*real(ifft2(FY2));XHAT=X1.*X2;
```

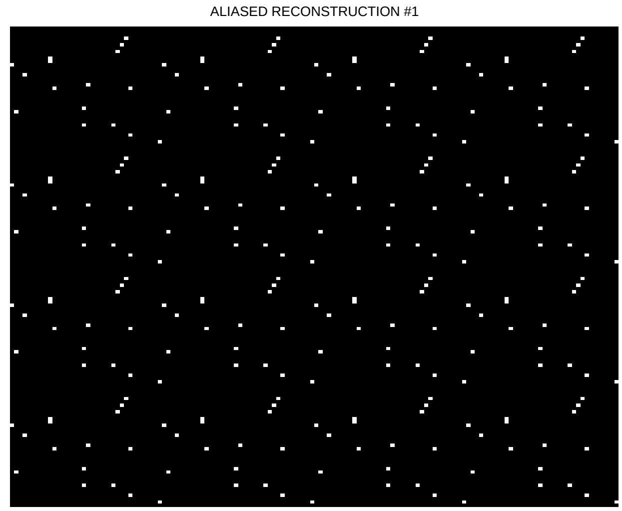


Fig. 1. Aliased reconstruction from downsampled DFT

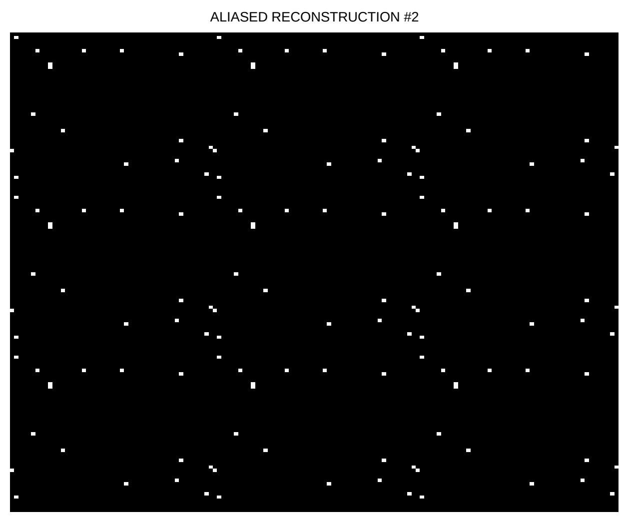


Fig. 2. Aliased reconstruction from downsampled DFT

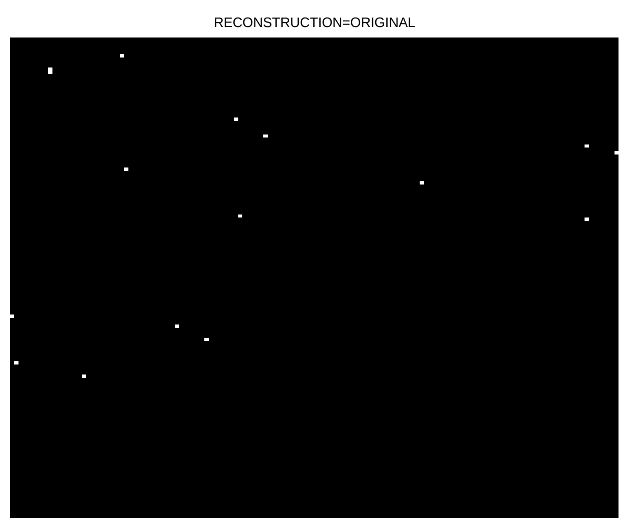


Fig. 3. Reconstructed matches original exactly

V. CLASSICAL DERIVATION

If S_{k_i}=1, then s_n is the *coherence* function C. For comparison, we present an alternate derivation which applies the "usual" derivation to the present problem.

A. Thresholding Algorithm for n_i

The known DFT values X_k of x_n are

$$X_{k_i} = \sum_{j=1}^{K} x_{n_j} e^{-j2\pi n_j k_i/N}, i = 1 \ldots M. \tag{25}$$

The estimates $\hat{x}_n$ of reconstructed x_n are

$$\hat{x}_n = \frac{1}{M}\sum_{i=1}^{M} X_{k_i} e^{j2\pi n k_i/N}, n = 0 \ldots N-1. \tag{26}$$

Substituting the first equation in the second gives

$$\hat{x}_n = \frac{1}{M}\sum_{i=1}^{M}\sum_{j=1}^{K} x_{n_j} e^{j2\pi k_i(n-n_j)/N} \tag{27}$$

Exchanging the order of the summations gives

$$\hat{x}_n = \frac{1}{M}\sum_{j=1}^{K} x_{n_j} \sum_{i=1}^{M} e^{j2\pi k_i(n-n_j)/N} \tag{28}$$

Now let $n = n_{j_o}$ for some $j = j_o$. Then

$$\begin{aligned} \hat{x}_n &= x_{n_{j_o}} \frac{1}{M}\sum_{i=1}^{M} e^{j2\pi k_i(n_{j_o}-n_{j_o})/N} \\ &+ \sum_{\substack{j=1 \\ j\neq j_o}}^{K} x_{n_j} \frac{1}{M}\sum_{i=1}^{M} e^{j2\pi k_i(n-n_j)/N} \\ &= x_{n_{j_o}} + \frac{1}{M}\sum_{\substack{j=1 \\ j\neq j_o}}^{K} x_{n_j} \sum_{i=1}^{M} e^{j2\pi k_i(n-n_j)/N} \end{aligned} \tag{29}$$

So the result of the inverse DFT is $\hat{x}_n$=:

$$\begin{cases} x_{n_{j_o}} + \frac{1}{M}\sum_{\substack{j=1 \\ j\neq j_o}}^{K} x_{n_j} \sum_{i=1}^{M} e^{j2\pi k_i(n-n_j)/N} & n = n_{j_o} \\ 0 \quad + \frac{1}{M}\sum_{j=1}^{K} x_{n_j} \sum_{i=1}^{M} e^{j2\pi k_i(n-n_j)/N} & n \neq n_{j_o} \end{cases}$$

The two interference terms are not quite identical. Hence a simple threshold of $|\hat{x}_n|$ will work if

$$|x_{\mathrm{MIN}}| - KC|x_{\mathrm{MAX}}| > KC|x_{\mathrm{MAX}}| \tag{30}$$

which becomes the condition

$$K < \frac{1}{2C}\frac{|x_{\mathrm{MIN}}|}{|x_{\mathrm{MAX}}|}. \tag{31}$$

B. Convergence of POCS for x_{n_i}

To analyze convergence behavior, define

- X_k=DFT[x_n]=known for $k = k_i$;
- $\hat{X}_k^L$=DFT[$\hat{x}_n^L$] at L^{th} iteration;
- $e_n^L = \hat{x}_n^L - x_n$=error at L^{th} iteration;
- E_k^L=DFT[e_n^L]=DFT[error] at L^{th} iteration.

The effect of the L^{th} iteration in DFT domain is

$$\hat{X}_k^{L+1} = \begin{cases} \hat{X}_k^L & \text{for } k \neq k_i \\ X_k & \text{for } k = k_i \end{cases} \tag{32}$$

Subtracting X_k from this gives

$$E_k^{L+1} = \begin{cases} E_k^L & k \neq k_i \\ 0 & k = k_i \end{cases} = E_k^L - \begin{cases} 0 & k \neq k_i \\ E_k^L & k = k_i \end{cases} \tag{33}$$

Since locations n_i are correctly identified at each iteration, starting with the first, an inverse DFT yields

$$e_{n_i}^{L+1} = e_{n_i}^L - \frac{1}{M}\sum_{k=1}^{M}\sum_{j=1}^{K} e^{j2\pi k_k(n_i-n_j)/N} e_{n_j}^L. \tag{34}$$

Noting the cancellation when $j = i$ gives

$$e_{n_i}^{L+1} = -\frac{1}{M}\sum_{k=1}^{M}\sum_{\substack{j=1 \\ j\neq i}}^{K} e^{j2\pi k_k(n_i-n_j)/N} e_{n_j}^L. \tag{35}$$

which immediately leads to the bound

$$|e_{\mathrm{MAX}}^{L+1}| \leq (KC)|e_{\mathrm{MAX}}^L| \tag{36}$$

so that the error decreases geometrically as $(KC)^L$.

Recalling the interference is bounded by

$$|\hat{x}_n^1 - x_n| \leq KC|x_{\mathrm{MAX}}| \tag{37}$$

shows that the *relative* (dimensionless) error is

$$\frac{|\hat{x}_n^1 - x_n|}{|x_n|} \leq \frac{|\hat{x}_n^1 - x_n|}{|x_{\mathrm{MIN}}|} \leq KC\frac{|x_{\mathrm{MAX}}|}{|x_{\mathrm{MIN}}|}. \tag{38}$$

Then the relative error at the L^{th} iteration is

$$\frac{|\hat{x}_n^L - x_n|}{|x_n|} \leq \frac{|x_{\mathrm{MAX}}|}{|x_{\mathrm{MIN}}|}(KC)^L \leq (1/2)^L. \tag{39}$$

The smaller the coherence C, the more nonzero values K of x_n that can be handled, and the faster the POCS algorithm converges. A relative error less than $(\frac{1}{2})^{10} \approx 0.001$ can be achieved in ten iterations.

In practice, these bounds are very conservative. A major point of this work is that the coherence C can be computed very quickly (a single inverse DFT) for a given configuration of known frequencies, and the effect of varying the locations of known frequencies can be determined quickly, to optimize the locations.