Closed-Form Reconstruction of Sparse Signals from 3 Sets of Downsampled Fourier Values

Andrew E. Yagle
Department of EECS, The University of Michigan, Ann Arbor, MI 48109-2122

Abstract— **We present a simple closed-form algorithm for reconstructing a sparse signal from three different sets of downsampled discrete Fourier transform (DFT) values. The algorithm, which requires only three inverse DFTs, can be viewed as a dual of harmonic product spectrum, which is used for musical pitch detection. The ratio of number of observations to sparsity is proportional to the cube root of the signal length. A Matlab program and a small sparse image example are provided.**

Keywords— **Sparse reconstruction**
Phone: 734-763-9810. Fax: 734-763-1503.
Email: aey@eecs.umich.edu. **EDICS: 2-REST.**

I. INTRODUCTION

A. Problem Statement

The N-point DFT X_k of the length=N signal x_n is

$$X_k=\sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N}, k=0\ldots N-1. \quad (1)$$

- x_n is real-valued and K-sparse. This means:
- x_n=0 unless $n\in\{n_1\ldots n_K\}$ (the n_i are unknown).
- Nonzero values of x_n are randomly distributed among the possible locations $n=0,1,2\ldots N$–1.

The DFT X_k is known for three sets of values of k:

- $k\in\{L_1,2L_1,3L_1\ldots N-L_1\}$
- $k\in\{L_2,2L_2,3L_2\ldots N-L_2\}$
- $k\in\{L_3,2L_3,3L_3\ldots N-L_3\}$
- L_1,L_2,L_3 are all factors of N.
- L_1,L_2,L_3 are pairwise relatively prime.
- The DC value X_0=$\sum_{n=0}^{N-1}x_n$ is also known.

The goal is to recover the K-sparse signal x_n from its known DFT values X_k. Note half of the given DFT values are complex conjugates of the other half.

B. Problem Significance

Many signals and images of practical interest have a sparse representation in a wavelet basis. Computation of the wavelet transform can be viewed as convolutions with scaled wavelet and scaling basis functions, which becomes multiplication in the DFT domain. Hence the results of this paper apply to both sparse signals and wavelet-sparsifiable signals.

A common approach to sparse reconstruction is to compute the minimum ℓ_1 norm solution, perhaps by linear programming. If the DFT frequencies k_i are randomly chosen, and if enough of them are known, then it has been shown that the minimum ℓ_1 norm solution is in fact x_n. In many practical situations we do not have the luxury of choosing the k_i at random-they are pre-specified. And the number M of X_{k_i} required is $O(K\log N)$ (the exact number is unknown).

Other approaches include thresholded Landweber iteration and orthogonal matching pursuit. These are much faster computationally, but require more of the problem in order to compute x_n. Since our apporach is completely different from all of these, we refer the reader to the extensive literature on these methods.

The algorithm presented in this paper requires $O(KN^{1/3})$ observations of DFT values. This is more than the minimum ℓ_1 norm solution requires, but the computation is closed-form (three inverse DFTs) and the DFT frequencies need not be random here.

II. Presentation of Algorithm

A. Replacing Unknown X_k with Zeros

Given data $X_k, k\in\{0,L,2L,3L\ldots N–L\}$, define

$$Y_k=\begin{cases} X_k & \text{for } k=0,L,2L\ldots N-L\\ 0 & \text{otherwise}\end{cases} \quad (2)$$

The inverse N-point DFT of Y_k is then

$$\begin{aligned} y_n &= \frac{1}{N}\sum_{k=0}^{N-1} Y_k e^{j2\pi nk/N}\\ &= \frac{1}{N}\sum_{k_1=0}^{N/L-1} X_{k_1L}e^{j2\pi n(k_1L)/N}\\ &= \frac{1}{L}\sum_{i=0}^{L-1} x_{n+iN/L}. \end{aligned} \quad (3)$$

So inserting zeros for the missing values of X_k and computing the inverse N-point DFT gives L copies of the nonzero values of x_n, each shifted in n by an integer multiple of N/L.

B. Algorithm

Repeat the above procedure for each of the three sets of values of X_k. This gives

$$\begin{aligned} y_{1n} &= \sum_{i_1=0}^{L_1-1} x_{n+i_1 N/L_1}. \\ y_{2n} &= \sum_{i_2=0}^{L_2-1} x_{n+i_2 N/L_2}. \\ y_{3n} &= \sum_{i_3=0}^{L_3-1} x_{n+i_3 N/L_3}. \end{aligned} \tag{4}$$

Then define the indicator function

$$y_n = y_{1n} y_{2n} y_{3n}. \tag{5}$$

The i_j=0 terms in (4) show that $y_{n_i} \neq 0$. But y_n for $n \neq n_i$ will be zero unless all three of $y_{in} \neq 0$. If N is large and the nonzero x_n are randomly distributed, this is unlikely (see the next section). y_n is an indicator for locations $\{n_i\}$ of nonzero x_n.

This concept has some similarity to harmonic product spectrum, as used for musical pitch detection, except that the time and frequency domains are exchanged, and shifted copies of x_n are used instead of harmonics of musical pitch frequencies.

III. Analysis of Data Requirement

We analyze the relation between four numbers:

- N=number of unknown values of x_n.
- K=number of nonzero values of x_n.
- M=number of known DFT values of x_n.
- p=Pr[Incorrect extra locations of nonzero x_n].

To simplify the analysis, we assume that

$$L_1 \approx L_2 \approx L_3 \approx L. \tag{6}$$

An easy way to make the L_i pairwise relatively prime is to make the L_i consecutive integers, with median L_i even. Note that N can be zero-padded to the next largest product of three consecutive integers.

In the following, we specifically exclude the i_j=0 terms (and hence the n=n_i terms) in (4).

Since nonzero x_n are randomly distributed over $\{0, 1 \ldots N–1\}$, for each specific value of n we have

$$\begin{aligned} Pr[x_n \neq 0] &= K/N \\ Pr[y_{in} \neq 0] &= KL/N \\ Pr[y_n \neq 0] &= (KL/N)^3 \end{aligned} \tag{7}$$

since $y_n \neq 0$ only if all three of $\{y_{1n}, y_{2n}, y_{3n}\} \neq 0$.

For all $n \neq n_i$, we then have

$$\begin{aligned} Pr[y_n = 0] &= [1-(KL/N)^3]^N \\ &\simeq e^{-K^3L^3/N^2} \\ Pr[y_n \neq 0] &= 1 - e^{-K^3L^3/N^2} \\ Pr[y_n \neq 0] &= p \simeq K^3L^3/N^2. \end{aligned} \tag{8}$$

using the definition of e=2.71828 and N is large.

The number of observations M is

$$M = \frac{N}{L_1} + \frac{N}{L_2} + \frac{N}{L_3} \approx \frac{3N}{L}. \tag{9}$$

Combining these gives the relation

$$M = K3N^{1/3} p^{-1/3}. \tag{10}$$

For example, to attain p=0.001 requires M=$30N^{1/3}$.

The minimum ℓ_1 norm requires M=$O(K \log N)$, but also requires much more computation and also requires random frequencies k.

IV. Numerical Example

We present a small example of reconstruction of a 44-sparse 210×210 image from three sets of downsampled 2-D DFT values. The example is small so that each algorithm step can be seen. We have

$$N = 210^2 = 44100; K = 44; L_1 = 5; L_2 = 6; L_3 = 7.$$

The number of observations is

$$M = (\frac{210}{5})^2 + (\frac{210}{6})^2 + (\frac{210}{7})^2 = 3889. \tag{11}$$

This includes conjugate symmetric values.

Results are shown in the figures on the next page.

A. Matlab Program

```
%N1,N2,N3:pairwise relatively prime
clear;N1=5;N2=6;N3=7;N=N1*N2*N3;
X=rand(N,N);X(X<0.999)=0;%So X sparse
FX=fft2(X);%Get X from downsampled FX:
F1=FX(1:N1:N,1:N1:N);%Downsample by N1
F2=FX(1:N2:N,1:N2:N);%Downsample by N2
F3=FX(1:N3:N,1:N3:N);%Downsample by N3
%Solution:
G1(N,N)=0;G2(N,N)=0;G3(N,N)=0;
G1(1:N1:N,1:N1:N)=F1;Y1=real(ifft2(G1));
G2(1:N2:N,1:N2:N)=F2;Y2=real(ifft2(G2));
G3(1:N3:N,1:N3:N)=F3;Y3=real(ifft2(G3));
%Y=indicator function for nonzero X:
Y=Y1.*Y2.*Y3*N*N;Y(abs(Y)>0.001)=1;
figure,imagesc(X),colormap(gray)
figure,imagesc(Y),colormap(gray)
```

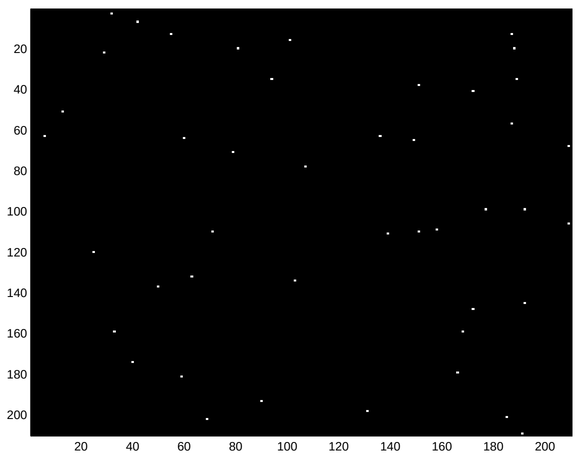

Fig. 1. Original 210×210 44-sparse image

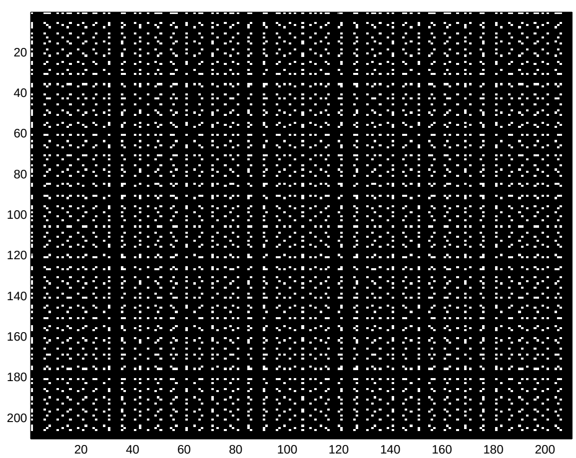

Fig. 2. Locations of 2-D DFT values used

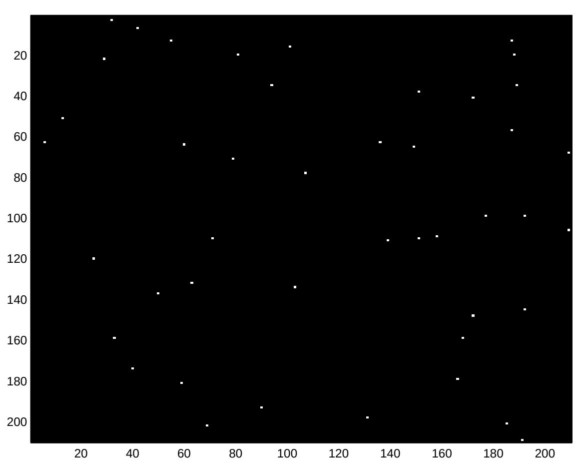

Fig. 3. Reconstructed 210×210 44-sparse image

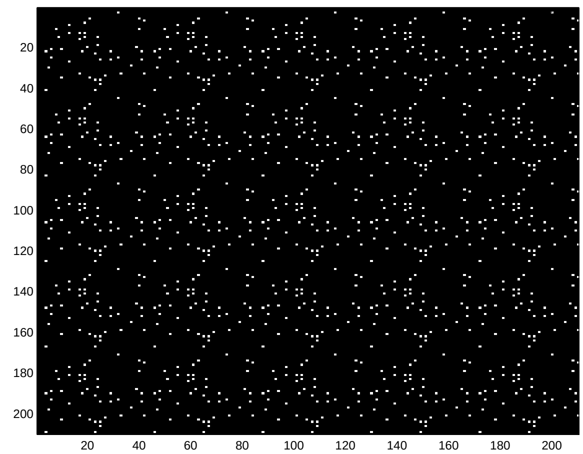

Fig. 4. Image from X_{k_1,k_2} using L_1=5

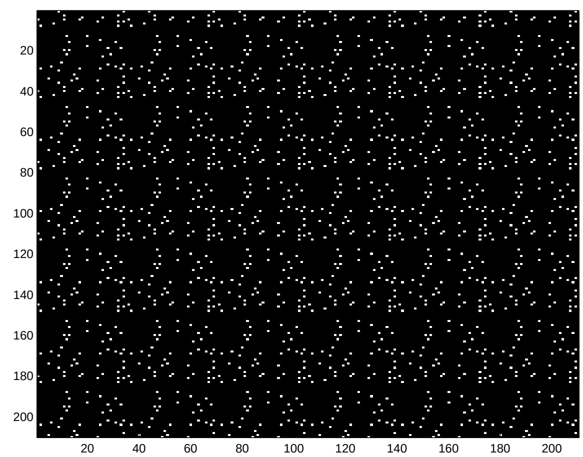

Fig. 5. Image from X_{k_1,k_2} using L_1=6

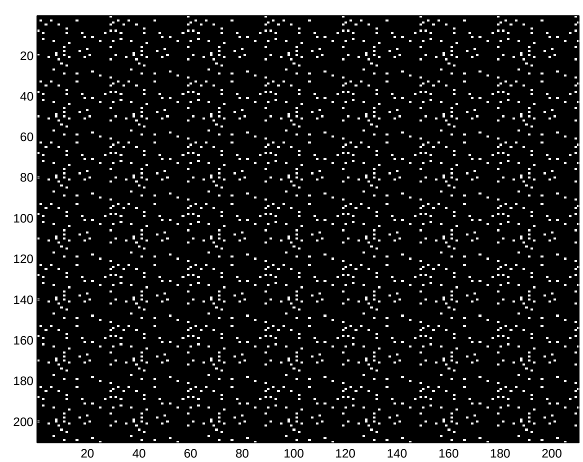

Fig. 6. Image from X_{k_1,k_2} using L_1=7