Closed-Form Reconstruction of Sparse Signals from 3 Sets of Downsampled Fourier Values

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Abstract—We present a simple closed-form algorithm for reconstructing a sparse signal from three different sets of downsampled discrete Fourier transform (DFT) values. The algorithm, which requires only three inverse DFTs, can be viewed as a dual of harmonic product spectrum, which is used for musical pitch detection. The ratio of number of observations to sparsity is proportional to the cube root of the signal length. A Matlab program and a small sparse image example are provided.

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I. INTRODUCTION

A. Problem Statement

The $N$-point DFT $X_k$ of the length-$N$ signal $x_n$ is

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N}, k = 0 \ldots N - 1.$$  \hspace{1cm} (1)

- $x_n$ is real-valued and $K$-sparse. This means:
- $x_n = 0$ unless $n \in \{n_1 \ldots n_K\}$ (the $n_i$ are unknown).
- Nonzero values of $x_n$ are randomly distributed among the possible locations $n = 0, 1, 2 \ldots N-1$.

The DFT $X_k$ is known for three sets of values of $k$:

- $k \in \{L_1, 2L_1, 3L_1 \ldots N - L_1\}$
- $k \in \{L_2, 2L_2, 3L_2 \ldots N - L_2\}$
- $k \in \{L_3, 2L_3, 3L_3 \ldots N - L_3\}$
- $L_1, L_2, L_3$ are all factors of $N$.
- $L_1, L_2, L_3$ are pairwise relatively prime.
- The DC value $X_0 = \sum_{n=0}^{N-1} x_n$ is also known.

The goal is to recover the $K$-sparse signal $x_n$ from its known DFT values $X_k$. Note half of the given DFT values are complex conjugates of the other half.

B. Problem Significance

Many signals and images of practical interest have a sparse representation in a wavelet basis. Computation of the wavelet transform can be viewed as convolutions with scaled wavelet and scaling basis functions, which becomes multiplication in the DFT domain. Hence the results of this paper apply to both sparse signals and wavelet-sparsifiable signals.

A common approach to sparse reconstruction is to compute the minimum $\ell_1$ norm solution, perhaps by linear programming. If the DFT frequencies $k_i$ are randomly chosen, and if enough of them are known, then it has been shown that the minimum $\ell_1$ norm solution is in fact $x_n$. In many practical situations we do not have the luxury of choosing the $k_i$ at random—they are pre-specified. And the number $M$ of $X_k$ required is $O(K \log N)$ (the exact number is unknown).

Other approaches include thresholded Landweber iteration and orthogonal matching pursuit. These are much faster computationally, but require more of the problem in order to compute $x_n$. Since our approach is completely different from all of these, we refer the reader to the extensive literature on these methods.

The algorithm presented in this paper requires $O(KN^{1/3})$ observations of DFT values. This is more than the minimum $\ell_1$ norm solution requires, but the computation is closed-form (three inverse DFTs) and the DFT frequencies need not be randomized here.

II. PRESENTATION OF ALGORITHM

A. Replacing Unknown $X_k$ with Zeros

Given data $X_k, k \in \{0, L, 2L, 3L \ldots N-L\}$, define

$$Y_k = \begin{cases} X_k & \text{for } k = 0, L, 2L \ldots N - L \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (2)

The inverse $N$-point DFT of $Y_k$ is then

$$y_n = \frac{1}{N} \sum_{k=0}^{N-1} Y_k e^{j2\pi nk/N}$$

$$= \frac{1}{N} \sum_{k_1=0}^{N/L-1} X_{k_1} e^{j2\pi(n(k_1L))/N}$$

$$= \frac{1}{L} \sum_{i=0}^{L-1} x_{n+iN/L}.$$  \hspace{1cm} (3)

So inserting zeros for the missing values of $X_k$ and computing the inverse $N$-point DFT gives $L$ copies of the nonzero values of $x_n$, each shifted in $n$ by an integer multiple of $N/L$.

B. Algorithm

Repeat the above procedure for each of the three sets of values of $X_k$. This gives
\[ y_{1n} = \sum_{i_1=0}^{L_1-1} x_{n+i_1 N}/L_1, \]
\[ y_{2n} = \sum_{i_2=0}^{L_2-1} x_{n+i_2 N}/L_2, \]
\[ y_{3n} = \sum_{i_3=0}^{L_3-1} x_{n+i_3 N}/L_3. \]  

Then define the indicator function
\[ y_n = y_{1n}y_{2n}y_{3n}. \]  

The \( i_j = 0 \) terms in (4) show that \( y_n, \neq 0 \). But \( y_n \) for \( n \neq n_i \) will be zero unless all three of \( y_{in} \neq 0 \). If \( N \) is large and the nonzero \( x_n \) are randomly distributed, this is unlikely (see the next section). \( y_n \) is an indicator for locations \( \{n_i\} \) of nonzero \( x_n \).

This concept has some similarity to harmonic product spectrum, as used for musical pitch detection, except that the time and frequency domains are exchanged, and shifted copies of \( x_n \) are used instead of harmonics of musical pitch frequencies.

III. Analysis of Data Requirement

We analyze the relation between four numbers:

- \( N \)=number of unknown values of \( x_n \).
- \( K \)=number of nonzero values of \( x_n \).
- \( M \)=number of known DFT values of \( x_n \).
- \( p \)=Pr[Incorrect extra locations of nonzero \( x_n \)].

To simplify the analysis, we assume that
\[ L_1 \approx L_2 \approx L_3 \approx L. \]  

An easy way to make the \( L_i \) pairwise relatively prime is to make the \( L_i \) consecutive integers, with median \( L_i \) even. Note that \( N \) can be zero-padded to the next largest product of three consecutive integers.

In the following, we specifically exclude the \( i_j = 0 \) terms (and hence the \( n=n_i \) terms) in (4).

Since nonzero \( x_n \) are randomly distributed over \( \{0, 1 \ldots N-1\} \), for each specific value of \( n \) we have
\[ Pr[x_n \neq 0] = K/N \]
\[ Pr[y_n \neq 0] = KL/N \]
\[ Pr[y_n \neq 0] = (KL/N)^3 \]  

since \( y_n \neq 0 \) only if all three of \( \{y_{1n}, y_{2n}, y_{3n}\} \neq 0 \).

For all \( n \neq n_i \), we then have
\[ Pr[y_n = 0] = [1 - (KL/N)^3]^N \approx e^{-K^3L^3/N^3} \]
\[ Pr[y_n \neq 0] = 1 - e^{-K^3L^3/N^3} \]
\[ Pr[y_n \neq 0] = p \approx K^3L^3/N^3. \]  

using the definition of \( e=2.71828 \) and \( N \) is large.

The number of observations \( M \) is
\[ M = \frac{N}{L_1} + \frac{N}{L_2} + \frac{N}{L_3} \approx \frac{3N}{L}. \]  

Combining these gives the relation
\[ M = K3N^{1/3}p^{-1/3}. \]  

For example, to attain \( p=0.001 \) requires \( M=30N^{1/3} \).

The minimum \( \ell_1 \) norm requires \( M=O(K \log N) \), but also requires much more computation and also requires random frequencies \( k \).

IV. Numerical Example

We present a small example of reconstruction of a 44-sparse \( 210 \times 210 \) image from three sets of downsampled 2-D DFT values. The example is small so that each algorithm step can be seen. We have
\[ N = 210^2 = 44100; K = 44; L_1 = 5; L_2 = 6; L_3 = 7. \]

The number of observations is
\[ M = (\frac{210}{5})^2 + (\frac{210}{6})^2 + (\frac{210}{7})^2 = 3889. \]

This includes conjugate symmetric values.

Results are shown in the figures on the next page.

A. Matlab Program

```
%N1,N2,N3:pairwise relatively prime
clear;N1=5;N2=6;N3=7;N=N1*N2*N3;
x=rand(N,N);X(x<0.999)=0;%So X sparse
FX=fft2(X);%Get X from downsampled FX:
F1=FX(1:N1,1:N1);%Downsample by N1
F2=FX(1:N2,1:N2);%Downsample by N2
F3=FX(1:N3,1:N3);%Downsample by N3

%Solution:
G1(N,N)=0;G2(N,N)=0;G3(N,N)=0;
G1(1:N1,1:N1)=F1;Y1=real(ifft2(G1));
G2(1:N2,1:N2)=F2;Y2=real(ifft2(G2));
G3(1:N3,1:N3)=F3;Y3=real(ifft2(G3));
Y=Y1.*Y2.*Y3.*N*N;Y(abs(Y)>0.001)=1;
figure,imagesc(X),colormap(gray)
figure,imagesc(Y),colormap(gray)
```
Fig. 1. Original 210×210 44-sparse image

Fig. 2. Locations of 2-D DFT values used

Fig. 3. Reconstructed 210×210 44-sparse image

Fig. 4. Image from $X_{k_1, k_2}$ using $L_1=5$

Fig. 5. Image from $X_{k_1, k_2}$ using $L_1=6$

Fig. 6. Image from $X_{k_1, k_2}$ using $L_1=7$