Divide-and-Conquer Image Reconstruction from Irregular DTFT Samples using Subband Decomposition and Gabor Logons

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Abstract— **The problem of reconstructing images from irregular frequency samples arises in synthetic aperture radar (SAR), magnetic resonance imaging (MRI), limited angle tomography, and 2-D filter design. Since there is no 2-D Lagrange interpolation formula, this problem is usually solved using an iterative algorithm such as POCS (Projection On Convex Sets) or CG (Conjugate Gradient) applied to a linear system of equations with the image pixels as unknowns. However, these require many iterations, each requiring a non-uniform forward 2-D Discrete Fourier Transform (DFT). We present a non-iterative algorithm for the reconstruction of an $(M \times M)$ image from a sufficient number of 2-D Discrete-Time Fourier-Transform (DTFT) samples. The algorithm uses Gabor logons, localized in space and wavenumber, to partition the problem into a set of smaller problems (divide-and-conquer), each of which is solved and then combined into the final reconstruction. The algorithm also can be used to obtain a low-resolution but unaliased reconstruction, and to regularize the problem by discarding sub-problems that are themselves ill-conditioned.**

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I. INTRODUCTION

A. Problem Statement

The goal is to reconstruct an $(M \times M)$ discrete image $x(i_1, i_2)$ from *some* of the values of its 2-D Discrete-Time-Fourier-Transform (DTFT)

$$X(\omega_1, \omega_2) = \sum_{i_1=0}^{M-1} \sum_{i_2=0}^{M-1} x(i_1, i_2) e^{-j2\pi(\omega_1 i_1 + \omega_2 i_2)}. \quad (1)$$

We formulate the problem for square image support; modification to non-square support is trivial. We assume that the frequency locations are already known.

B. Problem Discussion

If the $(M \times M)$ 2-D DFT is known *everywhere*, then clearly we can recover the $(M \times M)$ image using an inverse 2-D DFT. But if the DFT is only known for *some* of these DFT values, this will not work. While interpolation can be used to resample the frequency values to a rectangular lattice, this necessarily involves some approximation and some computation.

The problem of reconstructing an image from its irregular frequency samples arises in many applications. Some examples and the corresponding locations where the DTFT is known:

- *SAR:* On several arcs of points;
- *CAT:* On a polar raster of points;
- *MRI:* On a polar raster of points;
- *fMRI:* On a square spiral of points;
- *Limited-angle tomography:* In a bowtie region;
- *2-D FIR filter design:* A prescribed response.

C. Previous Approaches

One approach is to use Projection Onto Convex Sets (POCS). POCS alternately projects onto the spatial domain (imposing finite $(M \times M)$ support) and onto the DFT domain (imposing the known DFT values). While this algorithm is guaranteed to converge [1], there are several difficulties:

- Convergence requires many iterations, and algorithms cannot be parallelized in iteration;
- The forward DTFT must be computed at each iteration; this is a major problem in its own right. Using an interpolated DFT [2],[3] helps;
- Roundoff error in DFTs over many iterations may lead to problems in poorly conditioned problems, since the inverse DFT is not an exact inverse to the forward DFT in a finite-precision environment.

In [4] it is noted that, "Since the computational cost of the POCS method is several orders of magnitude higher than other methods, and it provides only poor rate of convergence, it is not included in our comparison" (slightly edited).

Another approach is to compute the least-squares solution by solving the $(M^2 \times M^2)$ linear system

$$(V^H V)x = V^H b \quad (2)$$

where
- V is a 2-D Vandermonde matrix;
- b is a vector of known DFT values;
- x is a vector of unknown pixel values.

The matrix $V^H V$ has Toeplitz-block-Toeplitz (TBT) structure [4],[5]. TBT matrix multiplication can be implemented using $(2M \times 2M)$ 2-D DFTs, suggesting the use of a preconditioned conjugate gradient (CG) algorithm. However, the following problems arise:

- Preconditioned CG still requires many iterations;
- Each iteration still requires three $(2M \times 2M)$ 2-D DFTs to implement the convolution;
- Computing $V^H b$ requires an $(N \times N)$ 2-D DFT;
- The condition number of $V^H V$ is the *square* of the condition number of V. Since V is almost surely already ill-conditioned, the noise amplification will be enormous unless drastic regularization is used;
- The number of CG iterations increases (roughly) with the condition number. Preconditioning helps, but many iterations will still be required.

Other approaches unwrap the 2-D problem into 1-D problems using either the Good-Thomas FFT [5] or variable substitution [6],[7]. This allows the use of the Lagrange interpolation formula, or a faster 1-D interpolation algorithm. An advantage of [5] is that given DFT values can lie anywhere on a rectangular lattice. The computation required has order $N^2 \log^2 N$.

D. New Approach

The approach used in this paper summarizes as follows (more details are provided in the next section):

1. Project the image onto an over-complete set of Gabor logons, with some overlap between subbands;
2. Each projection yields a self-contained finite-support image reconstruction subproblem from irregular 2-D DTFT samples, which are now in a subband of the original problem;
3. Each of these subproblems is solved separately, using any procedure. Any badly conditioned problem can be regularized or even discarded;
4. The 2-D DFT of the solution to each subproblem is computed, and the 2-D DFT of the Gabor logon divided out where the latter is not close to zero;
5. The computed 2-D DFTs for each subband are combined to give the 2-D DFT of the original image, which is then computed with an inverse 2-D DFT.

Advantages of the new approach are as follows:

- A large problem is replaced with many smaller and similar problems (divide-and-conquer);
- Each subproblem can be regularized independently, depending on its conditioning. Poorly conditioned or underdetermined subproblems (not enough frequency samples in that subband) can be discarded altogether, regularizing the overall problem;
- An *unaliased* low-resolution image can be reconstructed using the lowest-frequency subband. This may be sufficient for recognition in some applications.

II. NEW ALGORITHM

A. Gabor Logon

A Gabor logon is essentially a modulated Gaussian

$$\phi(t, t_o, \omega_0) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(t-t_o)^2/(2\sigma^2)} e^{j\omega_o t} \qquad (3)$$

Its exponential-squared dropoff in both time t and frequency ω means that the Gabor logon has virtually compact support in both time and frequency centered in time at $t = t_o$ and frequency at $\omega = \omega_o$. A Gaussian is used since it is most heavily concentrated in time and frequency; a parameter σ^2 trades off concentration in time and frequency.

Gabor logons are common choices for window function in the Short-Time Fourier Transform (STFT). However we are NOT peforming a time-frequency decomposition of the image reconstruction from irregular Fourier samples problem, since no explicit time-frequency representation interpretation is necessary.

Here we make the following changes to the Gabor logon basis function:

- We use discrete-time n instead of continuous-time;
- We use DTFT instead of Fourier transform to define its spectrum;
- We *truncate* in time n so that it has finite support;
- Despite these changes the spectrum has essentially compact support.

Empirically we have observed that the basis function

$$\phi(n, k) = \begin{cases} 0.9^{n^2/k} & \text{for } |n| \leq 6\sqrt{k} \\ 0 & \text{for } |n| > 6\sqrt{k} \end{cases} \qquad (4)$$

works well as a $(1/2)^k$-band filter (e.g., $k = 0 \rightarrow$ half-band filter). Note that since its dropoff is exponential-squared in time, the duration of basis function $\phi(n, k)$ does NOT increase by a factor of k, but by $\sqrt{k}$. This means that the procedure of this paper is more efficient for large k. Also note that $\phi(n, k)$ has duration $12\sqrt{k} + 1$.

B. Subband Subproblems

The 2-D DTFT of the projection having compact $(N \times N)$ support (here $**$ denotes 2-D convolution)

$$y(i_1, i_2) = x(i_1, i_2) ** \phi(i_1, k)\phi(i_2, k) \qquad (5)$$

is essentially zero except for frequencies

$$|\omega_1|, |\omega_2| < \pi(1/2)^k \tag{6}$$

Downsampling $y(i_1, i_2)$ by k leads to the following *subband subproblem*:

- Reconstruct $N \times N$ subimage $y(ki_1, ki_2)$, where
- $N = (M + 12\sqrt{k})/2^k$ from
- Frequency samples $X(\omega_1, \omega_2)\Phi(\omega_1)\Phi(\omega_2)$;
- all in the original subband $|\omega_1|, |\omega_2| < \pi(1/2)^k$;
- now expanded to the full band $|\omega_1|, |\omega_2| < \pi$.

For example, let the original image is (128×128). Using $k = 3$, we decompose the original problem into $2^{k^2} = 2^{3^2} = 64$ subproblems, each requiring reconstruction of an $(N \times N)$ subimage where

$$N = (128 + 12\sqrt{8})/2^3 = 20.2 \rightarrow N = 21 \tag{7}$$

from its 2-D DTFT frequency samples

$$X(8\omega_{1i}, 8\omega_{2i})), |\omega_1|, \omega_2| < \pi/8 \tag{8}$$

in a subband of the original problem. This requires the solution of 64 linear systems with $21^2 = 441$ unknowns, instead of one with $128^2 = 16384$ unknowns.

C. Complete Procedure

1. Define 2^{k^2} subproblems as above, using *modulation* either to shift the subproblem to the origin, or to shift the basis function to the frequency subband;
2. Solve each subproblem separately;
3. Do this *in parallel* for each of the 2^{k^2} subproblems;
4. Compute an $(N \times N)$ 2-D DFT of the solution;
5. Divide this point-by-point by the 2-D DFT of the *downsampled* $\phi(n, k)$
6. Combine these 2^{k^2} 2-D DFTs (one for each of 2^{k^2} subbands) into the 2-D DFT of the overall image;
7. Use *overlapping* subbands so frequencies at which $\phi(n, k)$ is small can be discarded.

III. EXAMPLE

The image "Lena" was downsampled for the (128×128) original image. Its (512×512)-point 2-D DFT was computed, and sampled irregularly. To make the computational savings even more dramatic, the sampling was chosen to be separable, but *differently separable*, within each of the 64 (64×64) subbands.

As noted above, each subproblem was to reconstruct a (21×21) image from samples within a (64×64) subband. Separable sampling used 22 out of 64 frequency values in each of the two subband dimensions. This yielded 44 systems of 22 equations in 21 unknowns for each subproblem (a (21×21) matrix inversion and two matrix multiplications).

The unaliased low-resolution reconstruction from the lowest subband is shown.

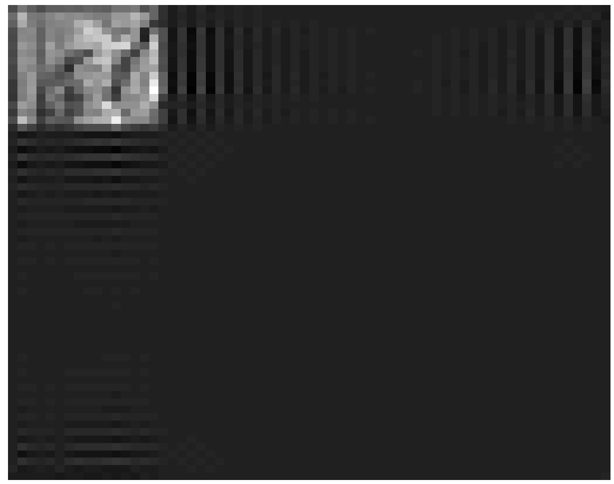

Fig. 1. Low-resolution reconstruction

```
load lena.mat;X=xx(1:2:256,1:2:256);FX=fft2(X,512,512);
G=0.993.^([-20:20].^2);GG=G'*G;FGG=fft2(GG,512,512);
FY=FX.*FGG;FG21=FGG([1:33 482:512],[1:33 482:512]);
K=[2 4 7 8 9 13 17 20 24 28 31];
A=exp(-2j*pi*([K 66-fliplr(K)]-1)'*[0:20]/64);
Y4H=A\FY(:,[K 514-fliplr(K)]).';
Y4HAT=A\Y4H(:,[K 514-fliplr(K)]).';
Y=ifft2(FY);Y4=64*Y(1:8:168,1:8:168);
X4HAT=ifft2(fft2(Y4HAT,64,64)./FG21);
imagesc(real(X4HAT(1:32,1:32))),colormap(gray),axis off
```

Student Version of MATLAB

IV. ACKNOWLEDGMENT

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