Fast Non-Iterative Image Reconstruction from Irregular 2-D DFT Frequency Samples

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Abstract— **The problem of reconstructing an image from irregular frequency samples arises in synthetic aperture radar (SAR), magnetic resonance imaging (MRI), limited angle tomography, and 2-D filter design. Since there is no 2-D Lagrange interpolation formula, this problem is usually solved using an iterative algorithm, such as POCS (Projection Onto Convex Sets), or CG (Conjugate Gradient) applied to a linear system of equations with the image pixels as unknowns. However, these require many iterations, and each iteration requires a non-uniform forward 2-D Discrete Fourier Transform (DFT). We present a non-iterative algorithm for the reconstruction of an $(M \times M)$ image from a sufficient number of arbitrary samples of its $(N \times N)$ 2-D DFT, where $N >> M$. The algorithm requires only a single sparse $(N \times N)$ 2-D DFT, followed by two roughly $(M \times M)$ 2-D DFTs. Precomputation for a given configuration of irregular $(N \times N)$ 2-D DFT samples is also required. Small and large examples illustrate the algorithm.**

Keywords— **Image reconstruction. EDICS: 2-rest**
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I. INTRODUCTION

A. Problem Statement

The goal is to reconstruct a real-valued $(M \times M)$ discrete image $x(i_1, i_2)$ from *some* of the values of its $(N \times N)$ 2-D DFT

$$X(k_1, k_2) = \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} x(i_1, i_2) e^{-j\frac{2\pi}{N^2}(i_1 k_1 + i_2 k_2)}. \quad (1)$$

We formulate the problem for square image support and DFT size; modification to the rectangular cases is trivial. We assume that the given DFT values are in complex conjugate pairs and that the frequency locations are already known.

B. Problem Discussion

If the $(N \times N)$ 2-D DFT is known *everywhere*, then clearly we can recover the $(M \times M)$ image using an inverse 2-D DFT. But if the DFT is only known for *some* of these DFT values, this will not work. While interpolation can be used to resample the frequency values to a rectangular lattice, this necessarily involves some approximation and some computation.

The problem of reconstructing an image from its irregular frequency samples (arbitrary values of its $(N \times N)$ 2-D DFT where N is large) arises in many applications. Some examples and the corresponding locations where the DFT is known:

- *SAR:* On several arcs of points;
- *CAT:* On a polar raster of points;
- *MRI:* On a polar raster of points;
- *fMRI:* On a square spiral of points;
- *Limited-angle tomography:* In a bowtie region;
- *2-D FIR filter design:* A prescribed response.

C. Previous Approaches

One approach is to use Projection Onto Convex Sets (POCS). POCS alternately projects onto the spatial domain (imposing finite $(M \times M)$ support) and onto the DFT domain (imposing the known DFT values). While this algorithm is guaranteed to converge [1], there are several difficulties:

- Convergence requires many iterations, and algorithms cannot be parallelized in iteration;
- The $(N \times N)$ DFT must be computed at each iteration. Even a pruned DFT requires much computation. Using an interpolated DFT [2],[3] helps;
- Roundoff error in DFTs over many iterations may lead to problems in poorly conditioned problems, since the inverse DFT is not an exact inverse to the forward DFT in a finite-precision environment.

In [4] it is noted that, "Since the computational cost of the POCS method is several orders of magnitude higher than other methods, and it provides only poor rate of convergence, it is not included in our comparison" (slightly edited).

Another approach is to compute the least-squares solution by solving the $(M^2 \times M^2)$ linear system

$$(V^H V)x = V^H b \quad (2)$$

- V is a 2-D Vandermonde matrix;
- b is a vector of known DFT values;
- x is a vector of unknown pixel values.

The matrix $V^H V$ has Toeplitz-block-Toeplitz (TBT) structure [4],[5]. TBT matrix multiplication can be implemented using $(2M \times 2M)$ 2-D DFTs, suggesting the use of a preconditioned conjugate gradient (CG) algorithm. However, the following problems arise:

- Preconditioned CG still requires many iterations;
- Each iteration still requires three $(2M \times 2M)$ 2-D DFTs to implement the convolution;
- Computing $V^H b$ requires an $(N \times N)$ 2-D DFT;
- The condition number of $V^H V$ is the *square* of the condition number of V. Since V is almost surely already ill-conditioned, the noise amplification will be enormous unless drastic regularization is used;
- The number of CG iterations increases (roughly) with the condition number. Preconditioning helps, but many iterations will still be required.

Other approaches unwrap the 2-D problem into 1-D problems using either the Good-Thomas FFT [5] or variable substitution [6],[7]. This allows the use of the Lagrange interpolation formula, or a faster 1-D interpolation algorithm. An advantage of [5] is that given DFT values can lie anywhere on a rectangular lattice. The computation required has order $N^2 \log^2 N$.

D. New Approach

The approach used in this paper summarizes as follows (more details are provided in the next section):

- Precompute, for a given configuration of DFT samples, a filter which has zero magnitude response at all unknown DFT sample locations;
- Compute the (sparse) $(N \times N)$ inverse 2-D DFT of the image whose frequency content is known (and filtered) and set to zero elsewhere;
- Deconvolve the original signal from this image and the filter. This requires two $(M \times M)$ 2-D DFTs;
- The total computation is at most $N^2 \log_2 N + 2M^2 \log_2 M$, or less if a sparse 2-D DFT is used.

II. NEW ALGORITHM

A. Filter Specification

Let Ω be the set of ordered pairs $\{(k_1, k_2)\}$ for which we know the $(N \times N)$ 2-D DFT $X(k_1, k_2)$ of the original image $x(i_1, i_2)$. Then define the 2-D filter with 2-D impulse response $h(i_1, i_2)$ and frequency response $H(k_1, k_2)$ as

$$\begin{aligned} h(i_1, i_2) &= 0 \quad for \quad N - M + 1 \leq i_1, i_2 \leq N - 1 \\ H(k_1, k_2) &= 0 \quad for \quad (k_1, k_2) \notin \Omega \\ H(k_1, k_2) &= \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} h(i_1, i_2) e^{-j \frac{2\pi}{N^2}(i_1 k_1 + i_2 k_2)} \end{aligned} \qquad (3)$$

Note that the nonzero values of $h(i_1, i_2)$ and $H(k_1, k_2)$ are known and determined (to an overall scale factor) by the above conditions.

The filter is *precomputed* for each configuration of known DFT values of interest. This is not a problem, since the frequency locations at which DFT values will be obtained are usually known in advance. So filters for all configurations of interest can be precomputed and stored. Storage is more efficient in the frequency domain, since fewer values of $H(k_1, k_2)$ than $h(i_1, i_2)$ are nonzero.

B. Filter Size

The size of the filter is determined as follows:

- $h(i_1, i_2)$ is $(N - M + 1) \times (N - M + 1)$;
- $h(i_1, i_2) = 0$ at $N^2 - (N - M + 1)^2$ points;
- $H(k_1, k_2) = 0$ at $(N - M + 1)^2$ points;
- $H(k_1, k_2)$ is arbitrary at $N^2 - (N - M + 1)^2$ points.

These numbers are specified as follows:

- $h(i_1, i_2) * x(i_1, i_2)$ must be $N \times N$;
- #unknowns=#equations in the linear system;
- $H(k_1, k_2)$ known at $N^2 - (N - M + 1)^2$ points.

Note that if $N >> M$, then

$$N^2 - (N - M + 1)^2 \approx 2NM = 2(\frac{N}{M})M^2 \qquad (4)$$

so that the original problem must be overdetermined by a factor of $2N/M$. This is not unreasonable; the existence of a unique solution to the original problem may require more frequencies than the number of image pixels [5]-[7]. Overdetermination is required since the 2-D deconvolution into which the original problem is transformed is itself overdetermined.

The filter could be computed by solving a large linear system of equations, but POCS for this *off-line* computation requires less storage. POCS was used to determine the filters for the examples below. No non-uniqueness issues have been encountered in computing filters, but it is possible that the filters themselves must also be overdetermined. Since this means fewer frequencies would be required for the original problem, this could only help.

C. Filter Approach

Now consider the filtered signal

$$y(i_1, i_2) = \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} h(j_1, j_2) x(i_1 - j_1, i_2 - j_2) \qquad (5)$$

Then the $(N \times N)$ 2-D DFT $Y(k_1, k_2)$ of $y(i_1, i_2)$ is

$$Y(k_1, k_2) = \begin{cases} H(k_1, k_2) X(k_1, k_2) & \text{for } (k_1, k_2) \in \Omega; \\ 0 & \text{for } (k_1, k_2) \notin \Omega \end{cases} \qquad (6)$$

Then we may compute $x(i_1,i_2)$ by deconvolution of the known $h(i_1,i_2)$ from the known $y(i_1,i_2)$ computed from the known $Y(k_1,k_2)$ using an $(N\times N)$ inverse 2-D DFT. Note that $Y(k_1,k_2)$ is sparse since most of the $(N\times N)$ 2-D DFT values are zero.

Deconvolution can be accomplished by computing the inverse 2-D DFT of the quotient of the 2-D DFTs of $y(i_1,i_2)$ and $h(i_1,i_2)$. However, the 2-D DFT values of $h(i_1,i_2)$ used for deconvolution must all be nonzero. If N is an integer multiple of M, then an $(M\times M)$ 2-D DFT must NOT be used here. Instead, a DFT of slightly different order is used, so different unit circle locations are sampled. Since $M << N$, this is not a problem computationally.

III. Tiny Example

A. Problem Specification

Consider the problem of reconstructing a 3×3 "image" from its frequencies marked with an x below:

$$\begin{bmatrix} x & * & * & * & x & * & * & * & x \\ * & x & * & * & x & * & * & x & * \\ * & * & x & * & x & * & x & * & * \\ * & * & * & x & x & x & * & * & * \\ x & x & x & x & x & x & x & x & x \\ * & * & * & x & x & x & * & * & * \\ * & * & x & * & x & * & x & * & * \\ * & x & * & * & x & * & * & x & * \\ x & * & * & * & x & * & * & * & x \end{bmatrix} \tag{7}$$

In this frequency sampling pattern:

- x denotes locations of known frequencies;
- $*$ denotes locations of unknown frequencies;
- The origin of the frequency plane is the center;
- The leftmost and rightmost columns are identical;
- The top and bottom rows are also identical;
- This polar raster is a type used in tomography;
- $N=8; M=3; N^2-(N-M+1)^2=28$;
- 28 values of (8×8) 2-D DFT are known;
- 36 values of (8×8) 2-D DFT are unknown.

This can be regarded as a tiny example of a tomography problem, in which an image is reconstructed from its projections at four angles.

B. Filter

The 6×6 filter $h(i_1,i_2)$ is specified as follows:

$$h(i_1,i_2)=0 \quad for \quad 6\le i_1,i_2\le 7$$
$$H(k_1,k_2)=0 \quad for \quad (k_1,k_2)\notin\Omega$$
$$H(k_1,k_2)=\sum_{i_1=0}^{7}\sum_{i_2=0}^{7} h(i_1,i_2)e^{-j\frac{2\pi}{64}(i_1k_1+i_2k_2)} \tag{8}$$

Ω is the set of 28 locations marked with x above. These 36 linear equations in 36 unknowns can be solved either directly or by using POCS. The result (rounded off) is

$$h(i_1,i_2)=\begin{bmatrix} 1.1 & .79 & .56 & .56 & .79 & 1.1 \\ .79 & 1.3 & .84 & .84 & 1.3 & .79 \\ .56 & .84 & 1.4 & 1.4 & .84 & .56 \\ .56 & .84 & 1.4 & 1.4 & .84 & .56 \\ .79 & 1.3 & .84 & .84 & 1.3 & .79 \\ 1.1 & .79 & .56 & .56 & .79 & 1.1 \end{bmatrix} \tag{9}$$

Note that the filter could be specified using either:

- 36 nonzero values of $h(i_1,i_2)$;
- 28 nonzero values of $H(k_1,k_2)$.

For this tiny example there is no reason to prefer $H(k_1,k_2)$ specification. But for realistic examples the savings in storage can be substantial.

C. Solution

In particular, suppose we are given the (rounded) (8×8) 2-D DFT values $H(k_1,0\le k_2\le 4)=$

$$\begin{bmatrix} 45 & 36e^{-j0.9} & -6-15j & 7.5e^{j1.4} & 15 \\ 38e^{-j1.12} & 32e^{-j2.0} & * & * & * \\ -18-15j & * & -5+8j & * & * \\ 14e^{j1.9} & * & * & 2.5e^{j2.8} & * \\ 15 & * & * & * & 5 \\ 14e^{-j1.9} & * & * & 1.5e^{-j0.9} & * \\ -18+15j & * & 5+4j & * & * \\ 38e^{j1.12} & 30e^{j0.2} & * & * & * \end{bmatrix} \tag{10}$$

- $H(0,0)$ is now in the upper left corner;
- The polar raster pattern is thus altered;
- $H(k_1,5\le k_2\le 7)=H^*(8-k_1,8-k_2)$;
- With conjugates, 28 of 64 values are known;
- The 36 unknowns are designated with $*$.

Now compute the (8×8) inverse 2-D DFT of

$$\begin{bmatrix} 45 & 36e^{-j0.9} & -6-15j & 7.5e^{j1.4} & 15 \\ 38e^{-j1.12} & 32e^{-j2.0} & 0 & 0 & 0 \\ -18-15j & 0 & -5+8j & 0 & 0 \\ 14e^{j1.9} & 0 & 0 & 2.5e^{j2.8} & 0 \\ 15 & 0 & 0 & 0 & 5 \\ 14e^{-j1.9} & 0 & 0 & 1.5e^{-j0.9} & 0 \\ -18+15j & 0 & 5+4j & 0 & 0 \\ 38e^{j1.12} & 30e^{j0.2} & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

where again conjugate values have been omitted to save space. The inverse 2-D DFT itself is omitted

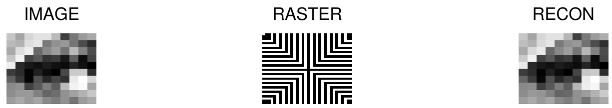

Fig. 1. Square-Raster Reconstruction of Eye

here. Then deconvolving the filter $h(i_1, i_2)$ above from this inverse 2-D DFT results in the image

$$x(i_1, i_2) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{12}$$

Note that even for this tiny example, the computation of an (8×8) inverse 2-D DFT and a small deconvolution is significantly smaller than that of solving a linear system of equations with nine unknowns.

IV. SMALL EXAMPLE

Consider the problem of reconstructing a 10×10 image of an eye from its frequency values on the concentric squares lattice shown in Figure 1 below.

This square raster is applicable to tomography; different sampling rates are used for different projections to produce this pattern. In this case the DFT transform size N need not be much larger than M, so that the tomography problem can be solved using three 2-D DFTs, each of size roughly $(M \times M)$.

V. ACKNOWLEDGMENT

This work was supported by DARPA and ARO under grant #DAAD-19-02-1-0262.

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