A simple non-iterative algorithm for 2-D tomography with unknown view angles

Andrew E. Yagle
Department of EECS, The University of Michigan, Ann Arbor, MI 48109-2122

Abstract— **Parallel-beam tomography in which the projection angles are unknown arises in MRI imaging, due to involuntary patient movement, and also in electron microscopic imaging of biological macromolecules, due to random orientation of many identical macromolecules. Subject to mild assumptions, uniqueness of the solution has been demonstrated. However, all previous algorithms for actually solving the problem have iteratively solved for both the unknown angles and the unknown image simultaneously, which is time-consuming and computationally intensive. We employ a new approach, based on the circular harmonic expansion, which decouples the angle estimation problem from the image estimation problem. We then show that the angle estimation problem can be transformed into a matrix eigenvalue problem of size equal to the number of angles, which is much less than the number of image pixels. Small and large numerical examples are provided.**

Keywords— **Image reconstruction; Tomography**
Phone: 734-763-9810. Fax: 734-763-1503.
Email: aey@eecs.umich.edu. **EDICS: 2-REST.**

I. INTRODUCTION

A. Overview

The basic parallel-beam tomography problem is to reconstruct an image from its projections at many different angles. These angles are often assumed to be equispaced, so that filtered backprojection can be used directly. Due to non-uniform rotation speed, the angles may not be equispaced, but they are at least assumed to be known.

However, there are situations in which the angles are *unknown.* In MRI, patient motion during long scan times can result in unknown angles. This is usually compensated by estimating the patient motion; however, this is not always possible. Another situation is electron microscopy of biological macromolecules (viruses), in which projection data at a single angle is taken for a large number of identical macromolecules at various random and unknown orientations. Assuming no overlapping for a single projection, so that individual particles can be distinguished, this is the same as having projection data at multiple but unknown angles.

Recently [1],[2] it has been shown that the projection (view) angles need *not* be known. Specifically, if projection data at more than 24 unknown and distinct view angles is known, a 2-D image is almost surely uniquely determined, to the obvious rotational ambiguity (Corollary 2 of Theorem 3 of [1]). This approach uses moments of the image and is quite complicated; we will not try to summarize it here.

This result was anticipated for the 3-D problem [3],[4]. In [3] a solution for the 3-D viral imaging problem noted above was presented. Uniqueness was demonstrated for the 3-D problem using the projection-slice theorem in [3] and [4]. However, this approach cannot be used for the 2-D problem [3].

Moments of the image have been used to reconstruct an unknown image from its projections at unknown angles in [1], [3] and [5]. The approach used here involves the Helgasson-Ludwig consistency conditions for the Radon transform, which involve the moments of the image. Conjectures made by the authors of [3] and [5] were discussed in detail in [1]. Other papers by the same authors will not be discussed here, to save space.

Reconstruction algorithms were proposed in [2]-[6]. However, all of these algorithms attempt to reconstruct simultaneously the unknown image and the unknown angles, which is a huge nonlinear problem. The algorithms are all iterative and require the solution of a large problem at each iteration.

B. New Approach

This paper uses the circular harmonic expansion (CHE) approach to image reconstruction from projections. The CHE is essentially a Fourier series expansion in the angular variable, which is of course periodic with period 2π. In the CHE domain, the Radon transform becomes a set of decoupled Abel transforms of various orders (see below). The CHE has been applied to image reconstruction from projections in [7]-[13], with considerable success.

The value of employing a CHE approach to the unknown view angles problem is that the problem of determining the angles *decouples* from the image reconstruction problem. That is, a small nonlinear problem is solved first, and the solution to that problem (the view angles) is used to solve a large linear problem (the image reconstruction problem). Of course, use of various estimation algorithms may result in nonlinear formulations for the angle determination

(AD) and image reconstruction (IR) problems, but decoupling them is still valuable in that it greatly reduces computation, regardless of the approach and algorithms used subsequently.

It should be emphasized that this is *not* the more general problem examined in [1] and [2]. Use of a truncated (finite series) CHP regularizes and parametrizes the problem. The problem is still of practical interest.

It should be noted that only the view angles are assumed to be unknown; an unknown translational shift in the projection data itself is not included. In the terminology of [1], this is the ARP problem, not the SHARP problem. However, this problem can be solved by translating the projection data so its collective center of mass is at the origin, and solving that problem for unknown angles ([1], page 1102).

This paper is organized as follows. Section II review the CHE and formulates the problem. Section III decouples the angle determination (AD) problem from the image reconstruction (IR) problem. Section III then transforms the AD problem into a small matrix eigenvalue problem. Section IV provides a tiny illustrative example of this procedure. Section V presents numerical examples. Section VI concludes with a summary and suggestions for future research.

II. CIRCULAR HARMONIC EXPANSION

A. Fourier Series Expansions

The projection $p(r,\theta)$, at view angle θ and distance r from the origin, of the image $x(\rho,\phi)$, expressed in polar coordinates, is defined as the *Radon transform*

$$p(r,\theta) = \int_0^\infty \int_0^{2\pi} x(\rho,\phi)\delta(r-\rho\cos(\theta-\phi))\rho\, d\phi\, d\rho \tag{1}$$

The goal of the parallel-beam tomography problem is to reconstruct the image $x(\rho,\phi)$ from a sampled set of its projections $\{p(r,\theta)\}$.

Since θ and ϕ are periodic with period 2π, both $p(r,\theta)$ and $x(\rho,\phi)$ are expanded in Fourier series

$$p(r,\theta) = \sum_{n=-\infty}^{\infty} p_n(r)e^{jn\theta} \tag{2}$$

$$x(\rho,\phi) = \sum_{n=-\infty}^{\infty} x_n(\rho)e^{jn\phi} \tag{3}$$

Here the n^{th} harmonics $p_n(r)$ and $x_n(\rho)$ of the projections and image, respectively, are defined as

$$p_n(r) = \frac{1}{2\pi}\int_0^{2\pi} p(r,\theta)e^{-jn\theta}d\theta \tag{4}$$

$$x_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} x(\rho,\phi)e^{-jn\phi}d\phi \tag{5}$$

These harmonics are related by [7]-[13]

$$\begin{aligned} p_n(r) &= \mathcal{A}_n\{x_n(\rho)\} \\ \mathcal{A}_n\{x_n(\rho)\} &= 2\int_r^\infty x_n(\rho)\frac{T_n(r/\rho)}{\sqrt{1-(r/\rho)^2}} \\ T_n(x) &= \cos(n\cos^{-1}x) \end{aligned} \tag{6}$$

where $T_n(x)$ is the Chebyschev polynomial of the first kind of order n.

B. Abel Transform Inversion

The n^{th}-order Abel transform $\mathcal{A}_n$ can be inverted in closed form using either of these two formulae:

$$x_n(\rho) = -\frac{1}{\pi}\int_\rho^\infty \frac{p'_n(r)T_n(r/\rho)}{\sqrt{r^2-\rho^2}}dr \tag{7}$$

$$\begin{aligned} x_n(\rho) &= \frac{1}{\pi}\int_0^\rho U_n(r/\rho)p'_n(r)dr \\ &-\frac{1}{\pi}\int_\rho^\infty \frac{\exp[-n\cosh^{-1}(r/\rho)]}{\sqrt{r^2-\rho^2}}p'_n(r)dr \\ U_n(x) &= \sin((n+1)\cos^{-1}x)/\sin(\cos^{-1}x) \end{aligned} \tag{8}$$

where $U_n(x)$ is the Chebyschev polynomial of the second kind of order n.

The first formula appeared in [7] and is called the "causal unstable" form in [10]. It properly reflects the causality property of the Radon transform: $\{x(\rho,\phi),\rho > R_o\}$ depends only on $\{p(r,\theta), r > R_o\}$. However, the integrand diverges as $\rho\to 0$ or $n\to\infty$.

The first formula was used in [13] to develop a causal state-space model for each $\mathcal{A}_n$, using a Brownian branch model as a prior for the harmonic. A Kalman *smoothing* filter then gave results in additive white Gaussian noise that were slightly superior to filtered backprojection.

The second formula appeared in [8] and [10] and is called the "noncausal stable" formula, since the integrands are bounded for large n. It was used in [10] to obtain good reconstructions, even in noise. It was also noted in [10] that filtered backprojection produces an image inconsistent with the original projections (i.e., reprojection does not produce the exact same projections), while the CHE produces a reconstructed image consistent with the projections.

The major point of the CHE formulation is that the reconstruction from projections problem *decouples* in harmonic order n. Using N harmonics and N discretization points for computation of $\mathcal{A}_n^{-1}$ permits inversion using $O(N^2\log N)$ *parallelizable* computations, since the inverse Abel transform formulae can be converted into convolutions using exponential radial sampling [10]-[13], so that the FFT can be used. This has been applied to region-of-interest tomography [12],[13], as exponential radial sampling is fine near the origin and coarse far away from the origin.

C. Problem Formulation

Let $x(\rho,\phi)$ be an image known to have:

- N nonzero circular harmonics $x_0(\rho)\dots x_{N-1}(\rho)$. Thus the image is angularly bandlimited. Note that $x_{-n}(\rho)=x_n^*(\rho)$ by conjugate symmetry, for a total of $(2N-1)$ real-valued functions of ρ;
- Compact support $0\le\rho\le R_o$, for some known R_o;
- Radial bandwidth such that $(2N-1)$ samples of ρ are sufficient to specify $x(\rho,\phi)$ for $0\le\rho\le R_o$;
- Linearly independent *sampled* harmonics.

We will not address the issue of radial sampling here, as many others have dealt with this subject.

Given projections (1) $p(r,\theta)$ of $x(\rho,\phi)$ for:

- $(2N+1)$ *unknown* angles θ_i, known to be distinct, such that no two differ by or add to π (otherwise they are not distinct);
- $(2N-1)$ *known* samples r_i of r. These samples need *not* be equidistant; exponential (or some quadrature) sampling may be used; for a total of:
- $(2N+1)(2N-1)$ total data points,

the goals are to:

- Reconstruct the unknown angles θ_i and
- Reconstruct the unknown image $x(\rho,\phi)$,
- Both to the unavoidable rotational ambiguity (any rotation of a solution will also be a solution).

The overall procedure will be to:

- Compute unknown angles θ_i (nonlinear problem);
- Compute the unknown projection harmonics $p_n(r)$ (a linear problem);
- Compute N independent Abel transforms (linear);
- Compute $x(\rho,\phi)$ from the image harmonics $x_n(\rho)$.

From a degrees-of-freedom perspective, we have:

- $(2N+1)(2N-1)$ equations (projections at $(2N+1)$ unknown view angles with $(2N-1)$ radial samples)
- In $(2N+1)+(2N-1)^2$ unknowns ($(2N+1)$ view angles, and $(2N-1)$ real-valued image harmonic functions with $(2N-1)$ radial samples each);
- So the problem is slightly overdetermined; $(2N)$ view angles would make it underdetermined by one.

Also note that the numbers of nonzero projection and image harmonic complex functions are both N.

III. UNKNOWN ANGLES COMPUTATION

A. New Formulation

Define the matrices

$$P=\begin{bmatrix} p(r_1,\theta_1) & \cdots & p(r_{2N-1},\theta_1)\\ \vdots & \cdots & \vdots\\ p(r_1,\theta_{2N+1}) & \cdots & p(r_{2N-1},\theta_{2N+1})\end{bmatrix}\tag{9}$$

$$V=\begin{bmatrix} e^{j(1-N)\theta_1} & \cdots & e^{j(N-1)\theta_1}\\ \vdots & \cdots & \vdots\\ e^{j(1-N)\theta_{2N+1}} & \cdots & e^{j(N-1)\theta_{2N+1}}\end{bmatrix}\tag{10}$$

$$H=\begin{bmatrix} p_{1-N}(r_1) & \cdots & p_{1-N}(r_{2N-1})\\ \vdots & \cdots & \vdots\\ p_{N-1}(r_1) & \cdots & p_{N-1}(r_{2N-1})\end{bmatrix}.\tag{11}$$

Truncating (2) to N nonzero harmonics yields

$$p(r,\theta)=\sum_{n=1-N}^{N-1}p_n(r)e^{jn\theta}\tag{12}$$

Then substituting

$$r=r_i,\quad i=1\dots(2N-1)$$
$$\theta=\theta_j,\quad j=1\dots(2N+1)$$

yields the matrix equation

$$P=VH\tag{13}$$

- P is known from the data;
- V and H are both unknown;
- V has Vandermonde structure.

The goal is to compute V and H from P.

Note that this is *not* the problem solved by Prony's method. Prony's method computes N unknown poles p_i and unknown residues c_i from a known segment of length $(2N+1)$ of a time series $y(n)$ by solving

$$\begin{bmatrix} y(0)\\ \vdots\\ y(2N)\end{bmatrix}=\begin{bmatrix} 1 & \cdots & 1\\ p_1^1 & \cdots & p_N^1\\ \vdots & \cdots & \vdots\\ p_1^{2N} & \cdots & p_N^{2N}\end{bmatrix}\begin{bmatrix} c_1\\ \vdots\\ c_N\end{bmatrix}\tag{14}$$

Note that the matrix in Prony's method is the transpose of a Vandermonde matrix (some reverse the definitions we have used of a Vandermonde matrix and its transpose). Either way, we cannot use Prony's method or array processing algorithm that solves the same problem, to solve the present one.

B. Angle Computation

One way to solve our problem is to compute the two left null vectors N_1^T and N_2^T of the matrix P (these exist since P is $(2N+1)\times(2N-1)$). Then

$$[0] = N_i^T P = N_i^T V H \rightarrow [0]H^{-1} = [0] = N_i^T V \quad (15)$$

assuming H has full rank. This yields $(4N-2)$ simultaneous polynomial equations in $(2N+1)$ unknowns $e^{j\theta_i}$. By Bezout's theorem, this overdetermined polynomial system will almost surely not have more than the one solution it is known to have.

Note this also decouples determination of the angles from determination of the image, as desired. Also note that $(2N)$ angles, instead of $(2N+1)$, would result in an underdetermined problem, since there will only be a single null vector, yielding $(2N-1)$ equations in $(2N)$ unknowns.

However, there is a better way, as we now demonstrate. Define the matrices

$$\tilde{P} = \begin{bmatrix} p(r_1,\theta_1) & \cdots & p(r_{2N-1},\theta_1) & 0 \\ \vdots & \cdots & \cdots & \vdots \\ p(r_1,\theta_{2N-1}) & \cdots & p(r_{2N-1},\theta_{2N-1}) & 0 \\ p(r_1,\theta_{2N+0}) & \cdots & p(r_{2N-1},\theta_{2N+0}) & \lambda \\ p(r_1,\theta_{2N+1}) & \cdots & p(r_{2N-1},\theta_{2N+1}) & \mu \end{bmatrix} \quad (16)$$

$$\tilde{V} = \begin{bmatrix} e^{j(1-N)\theta_1} & \cdots & e^{jN\theta_1} \\ \vdots & \cdots & \vdots \\ e^{j(1-N)\theta_{2N+1}} & \cdots & e^{jN\theta_{2N+1}} \end{bmatrix} \quad (17)$$

$$\tilde{H} = \begin{bmatrix} p_{1-N}(r_1) & \cdots & p_{1-N}(r_{2N-1}) & q_0 \\ \vdots & \cdots & \cdots & \vdots \\ p_{N-1}(r_1) & \cdots & p_{N-1}(r_{2N-1}) & q_{2N-2} \\ 0 & \cdots & 0 & q_{2N-1} \end{bmatrix} \quad (18)$$

$$\begin{aligned} q(z) &= q_0 + q_1 z + \dots q_{2N-1} z^{2N-1} \\ 0 &= q(z = e^{j\theta_i}), i = 1 \dots (2N-1) \\ \lambda &= q(z = e^{j\theta_{2N}}); \quad \lambda \quad \text{unknown} \\ \mu &= q(z = e^{j\theta_{2N+1}}); \quad \mu \quad \text{unknown} \end{aligned} \quad (19)$$

Note that:

- $\tilde{P} = \tilde{V}\tilde{H}$ is true, as was $P = VH$;
- $\tilde{P}$ is known except for constants λ and μ;
- $\tilde{V}$ is the Vandermonde extension of V;
- $\tilde{H}$ is augmented by a row and column.

Now consider the effect of augmenting $\tilde{H}$ with any column but its last *circularly shifted down one.* Using the Vandermonde structure of $\tilde{V}$, this augments $\tilde{P}$ with its corresponding column with the n^{th} element multiplied by $e^{j\theta_n}$ (this is essentially the delay property of Fourier series).

This augmented $\tilde{P}$ has size $(2N+1)\times(2N+1)$, and it is the product of the $(2N+1)\times(2N)$ matrix $\tilde{V}$ and the $(2N)\times(2N+1)$ matrix (augmented) $\tilde{H}$. Hence the augmented $\tilde{P}$ matrix is singular, and has zero determinant. Computing this determinant and setting it equal to zero yields a *linear* equation in the unknown $e^{j\theta_i}$ (except for the unknown λ and μ).

Performing this in turn for each of the first $(2N-1)$ columns of $\tilde{P}$ yields $(2N-1)$ *linear* equations in $(2N+1)$ unknowns $e^{j\theta_i}$ (except for the unknown λ and μ). Augmenting these with the two linear equations obtained from the two left null vectors N_1^T and N_2^T results in a *generalized eigenvalue problem* with $\frac{\lambda}{\mu}$ as the generalized eigenvalue and the $e^{j\theta_i}$ as the elements of the generalized eigenvector.

There is still one difficulty remaining. It is clear that $\theta_n = 0$ will satisfy all of the equations noted so far, since setting $e^{j\theta_n} = 1$ means that all of the determinants have two identical columns, so they are all always zero. Unfortunately, the left null vectors will not help here, since one of the columns of the Vandermonde matrix consists of all ones. This is a most unfortunate coincidence. Hence we need one more equation for the $e^{j\theta_n}$. Of course, these are all known to have unit magnitude, and this can be used, but another linear equation would simplify matters.

IV. TINY EXAMPLE

To illustrate the above procedure for computing the angles, we present a tiny example. For simplicity we dispense with all of the Abel transform procedure, since this is well-established [7]-[13]. We also do not assume that the roots in the Vandermonde matrix lie on the unit circle; instead we assume only that their sum is zero (this is required to obtain an overdetermined problem).

A. Problem Statement

We are given known projections at six angles, sampled at four radii each. These are arranged into the (6×4) matrix P

$$P = \begin{bmatrix} 22 & 20 & 20 & 18 \\ -6 & -12 & -2 & 0 \\ 105 & 87 & 100 & 69 \\ -210 & -188 & -200 & -26 \\ -513 & -435 & -500 & -87 \\ 1278 & 996 & 1264 & 606 \end{bmatrix} \quad (20)$$

The goal is to determine a (6×4) Vandermonde matrix V and an arbitrary (4×4) matrix H such that $P = VH$.

B. Obtaining Linear Equations

Augmenting the matrices as discussed above, we obtain

$$\begin{bmatrix} 22\omega_1 & 22 & 20 & 20 & 18 & 0 \\ -6\omega_2 & -6 & -12 & -2 & 0 & 0 \\ 105\omega_3 & 105 & 87 & 100 & 69 & 0 \\ -210\omega_4 & -210 & -188 & -200 & -26 & 0 \\ -513\omega_5 & -513 & -435 & -500 & -87 & \lambda \\ 1278\omega_6 & 1278 & 996 & 1264 & 606 & \mu \end{bmatrix} = \begin{bmatrix} \omega_1^0 & \omega_1^1 & \omega_1^2 & \omega_1^3 & \omega_1^4 \\ \omega_2^0 & \omega_2^1 & \omega_2^2 & \omega_2^3 & \omega_2^4 \\ \omega_3^0 & \omega_3^1 & \omega_3^2 & \omega_3^3 & \omega_3^4 \\ \omega_4^0 & \omega_4^1 & \omega_4^2 & \omega_4^3 & \omega_4^4 \\ \omega_5^0 & \omega_5^1 & \omega_5^2 & \omega_5^3 & \omega_5^4 \\ \omega_6^0 & \omega_6^1 & \omega_6^2 & \omega_6^3 & \omega_6^4 \end{bmatrix} \times \begin{bmatrix} 0 & h_{11} & h_{12} & h_{13} & h_{14} & q_0 \\ h_{11} & h_{21} & h_{22} & h_{23} & h_{24} & q_1 \\ h_{21} & h_{31} & h_{32} & h_{33} & h_{34} & q_2 \\ h_{31} & h_{41} & h_{42} & h_{43} & h_{44} & q_3 \\ h_{41} & 0 & 0 & 0 & 0 & q_4 \end{bmatrix} \quad (21)$$

The (6×6) matrix is the product of (6×5) and (5×6) matrices, so it is singular. Setting its determinant equal to zero gives

$$\begin{aligned} &[-1164240\omega_1 - 352800\omega_2 + 2469600\omega_3 \\ &+11113200\omega_4 - 12065760\omega_5]\mu = \\ &[9313920\omega_1 + 1128960 + \omega_2 - 31610880\omega_3 \\ &-8890560\omega_4 + 30058560\omega_6]\lambda \end{aligned} \quad (22)$$

C. Generalized Eigenvalue Problem

Proceeding similarly with columns #2-4 gives three similar equations. Augmenting this with the two left null vectors of the known (6×4) matrix P, and a row of ones (since the sum of the ω_i is zero) yields the generalized eigenvalue problem (matrix entries are rounded)

$$\mu \begin{bmatrix} -.116 & -.035 & .247 & 1.111 & -1.21 & 0 \\ -.106 & -.071 & .205 & .995 & -1.02 & 0 \\ -.106 & -.012 & .235 & 1.06 & -1.18 & 0 \\ -.095 & 0 & .162 & .138 & -.205 & 0 \\ .769 & -.277 & -.57 & -.011 & .043 & .05 \\ 0 & .539 & .216 & -.73 & .360 & .05 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \end{bmatrix} = \lambda \begin{bmatrix} .931 & .113 & -3.16 & -.89 & 0 & 3.01 \\ .847 & .226 & -2.62 & -.80 & 0 & 2.34 \\ .847 & .038 & -3.01 & -.85 & 0 & 2.97 \\ .762 & 0 & -2.08 & -.11 & 0 & 1.43 \\ 0 & .540 & -.216 & -.73 & .36 & .045 \\ .769 & -.28 & -.573 & -.011 & .04 & .048 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \end{bmatrix} \quad (23)$$

Solving this generalized eigenvalue problem yields the six generalized eigenvalues

$$\frac{\lambda}{\mu} = [0, 1, -1, \frac{5}{32}, \frac{5}{32}, \frac{5}{32}] \quad (24)$$

There are always generalized eigenvalues at $\{0, \pm 1\}$; the others are all clustered at the actual value of $\frac{\lambda}{\mu}$. The generalized eigenvector associated with these is the actual solution

$$[\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6] = [1, -1, 2, -3, -4, 5] \quad (25)$$

to a scale factor.

The matrix H of sampled projection harmonics is

$$H = \begin{bmatrix} 3 & 1 & 4 & 1 \\ 5 & 9 & 2 & 6 \\ 5 & 3 & 5 & 8 \\ 9 & 7 & 9 & 3 \end{bmatrix} \quad (26)$$

which was constructed from the digits of π.

The polynomial $q(z)$ and actual values of λ and μ:

$$q(z) = 6 - z - 7z^2 + z^3 + z^4; \lambda = 90; \mu = 576 \quad (27)$$

D. Discussion

Note that there is an unavoidable scale factor ambiguity in computing the ω_i. In the unknown-angles tomography problem it is known that

$$|\omega_i| = |e^{\theta_i}| = 1 \quad (28)$$

and this scale-factor ambiguity implies that θ_i are only determined to an additive constant. This is the unavoidable rotational ambiguity noted earlier, arising automatically in our proposed procedure.

Also note that, due to algorithm error, the multiple eigenvalue at 5/32 is split slightly among several close values in a circle around the true value. Hence the true value of $\frac{\lambda}{\mu}$ can be found by averaging. Inserting this value into the generalized eigenvalue problem yields a unique eigenvector since the problem is overdetermined ((7×6) in this case).

Looking back at the problem statement, one may wonder where the Vandermonde structure of V was used. This is used implicitly when augmenting $\tilde{H}$ with one of its columns that has been circularly shifted. The effect of this on $\tilde{P}$ is to multiply the corresponding column point-by-point with $\{\omega_i\}$, due to the Vandermonde structure of $\tilde{V}$.

V. Examples

VI. Conclusion

VII. Acknowledgments

The work of the authors was supported by DARPA and ARO under grant #DAAD-19-02-1-0262.

References

[1] S. Basu and Y. Bresler, "Uniqueness of tomography with unknown view angles," *IEEE Trans. Image Proc.* 9(6), 1094-1106, 2000.

[2] S. Basu and Y. Bresler, "Feasibility of tomography with unknown view angles," *IEEE Trans. Image Proc.* 9(6), 1107-1122, 2000.

[3] A.B. Goncharov, "Three-dimensional reconstruction of arbitrarily arranged identical particles given their projections," in *Mathematical Problems in Tomography*, Am. Math. Soc., Providence RI, 1990, 67-96.

[4] P.D. Lauren and N. Nandhakumar, "Estimating the viewing parameters of random, noisy projections of asymmetric objects for tomographic reconstruction," *IEEE Trans. PAMI* 19(5), 417-430, 1997.

[5] D. Salzman, "A method of general moments for orienting 2D projections of unknown 3D objects," *CVGIP* 50, 129-156, 1990.

[6] P.A. Penczek, R. Renka, M. Schomberg, "Gridding-based direct Fourier inversion of the three-dimensional x-ray transform," *J. Opt. Soc. Am. A*, 21(4), 499-509, 2004.

[7] A.M. Cormack, "Representation of a function by its line integrals, with some radiological applications. I," *J. Appl. Phys.* 34, 2722-2727, 1963.

[8] E.W. Hansen, "Theory of circular harmonic image reconstruction," *J. Opt. Soc. Am.* 71, 304-308, 1981.

[9] W.G. Hawkins and H.H. Barrett, "A numerically stable circular harmonic reconstruction algorithm," *SIAM J. Num. Anal.* 23, 873-890, 1986.

[10] C.H. Chapman and P.W. Cary, "The circular harmonic Radon transform," *Inverse Problems* 2, 23-49, 1986.

[11] W.E. Higgins and D.C. Munson, "A Hankel transform approach to tomographic image reconstruction," *IEEE Trans. Med. Imag.* 7, 59-72, 1988.

[12] B. Sahiner and A.E. Yagle, "Region-of-interest tomography using exponential radial sampling," *IEEE Trans. Image Proc.* 4(8), 1120-1127, 1995.

[13] D.S. Luo and A.E. Yagle, "A Kalman filtering approach to stochastic global and region-of-interest tomography," *IEEE Trans. Image Proc.* 5(3), 471-479, 1996.