Engin. 100: Music Signal Processing
Lab #3[1]: Computing Signal Spectra

• How do we define the spectrum of a signal?
• Why can we sample without losing information?
• How do we compute the spectrum of a signal?
• How do we use Matlab to compute the spectrum?
• Why should we bother to compute the spectrum? (to be answered next week—we need to know how to compute spectra before we can use them!)

What is the spectrum of a signal?

• In Engin. 100, we will only define spectra of periodic (repeating) signals. How come?

1. Each musical note played by an instrument is a periodic signal, so this is what we care about.
2. Definition and computation are much easier.
3. Actually, a non-periodic signal can be viewed as a periodic signal with a VERY long period.

Fourier Series of Periodic Signals

• Any real-world periodic signal with period=T (so x(t)=x(t+T) for all t) can be expanded as:
  \[ x(t) = c_0 + c_1 \cos(2\pi t/T - \theta_1) + c_2 \cos(4\pi t/T - \theta_2) + \ldots \]
  
  - DC constant
  - fundamental period=T
  - harmonic period=T/2

  The spectrum of x(t) is just a stem plot of the amplitudes \( \{c_k\} \) vs. the frequencies \( \{k/T\} \) Hertz.

Fourier Series of Musical Signals

• Phase shift \( \theta \) cannot be perceived by the ear.
• No DC term in musical signals. So can use:
  \[ x(t) = a_1 \cos(2\pi t/T) + a_2 \cos(4\pi t/T) + a_3 \cos(6\pi t/T) + \ldots \]
  
  - fundamental
  - harmonic
  - harmonic

• This model originally proposed by Bernoulli.
• Coefficients \( a_k \) = timbre (TAM-ber) of sound:
  - Is why a clarinet sounds different from a flute.
  - Plot of \( a_k \) vs. frequency \( k/T \) Hertz: sound spectrum.
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Fourier Series of Periodic Signals

• Any real-world periodic signal with period=T can be expanded in either of these two forms:

\[
\begin{align*}
 x(t) &= c_0 + c_1 \cos(2\pi t/T - \theta_1) + c_2 \cos(4\pi t/T - \theta_2) + \ldots \\
 x(t) &= a_0 + a_1 \cos(2\pi t/T) + a_2 \cos(4\pi t/T) + \ldots \quad + b_1 \sin(2\pi t/T) + b_2 \sin(4\pi t/T) + \ldots
\end{align*}
\]

• Coefficients in these two forms are related by:
  \[ A \cos(t) + B \sin(t) = C \cos(t - \theta) \] (cf. Lab #1) where \( C = (A^2 + B^2)^{1/2}; \tan \theta = B/A; A = C \cos \theta; B = C \sin \theta. \)

Example: Period=0.001 seconds

• x(t) has period=T=0.001 sec. x(t) has expansion:

\[
\begin{align*}
 x(t) &= a_0 + a_1 \cos(2\pi 1000 t) + a_2 \cos(2\pi 2000 t) + \ldots \\
 &= b_1 \sin(2\pi 1000 t) + b_2 \sin(2\pi 2000 t) + \ldots
\end{align*}
\]

• 0 1000 Hz (fund.) 2000 Hz (harmonic)

• In general, this expansion is infinite and requires infinitely high frequencies at integer multiples of the fundamental frequency=1/period=1000 Hz.
• But what if the signal is bandlimited to 8000 Hz?

Sampling Signals with Period=T
seconds and Bandlimited to B Hertz

• IDEA: If a signal is completely characterized by 2BT+1 numbers, we can compute those 2BT+1 numbers by sampling 2BT+1 times per period!

• THAT IS: Sample \textit{faster} than 2BT times/period.

• Sampling rate>(2BT times)/(T second)=2B Hertz. \textit{Need:} sampling rate>2(maximum frequency).

Sampling theorem: discovered by Claude Shannon, U-M CoE Class of 1936 (EE). Bust outside EECS.
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Easiest way to illustrate how to do this:

Use a small numerical example

- \(x(t)\) is periodic with period \(T=0.001\) second.
- \(x(t)\) has maximum frequency \(B=1000\) Hertz.
- \(x(t)\) is then sampled at 4000 samples/second.
- Need: \(2BT+1=2(1000)(0.001)+1=3\) numbers.

\[
\begin{align*}
[x(t)] &= a_0 + a_1 \cos(2\pi t/0.001) + b_1 \sin(2\pi t/0.001). \\
\text{Sampling: } &\text{Have } x(t)=x(n/4000) \text{ for } n=0,1,2,3 \text{ since } x(4/4000)=x(0.001)=x(0) \text{ again (periodic).}
\end{align*}
\]

Small Numerical Example, Continued

- We observe these samples:

<table>
<thead>
<tr>
<th>(n/4000)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x(n/4000))</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>
- \(x(1/4000)=a_0+a_1\cos(\pi/2)+b_1\sin(\pi/2)=a_0+b_1=4\).
- \(x(2/4000)=a_0+a_1\cos(\pi)+b_1\sin(\pi)=a_0-a_1=1\).
- \(x(3/4000)=a_0+a_1\cos(3\pi/2)+b_1\sin(3\pi/2)=a_0-b_1=2\).
- \(x(4/4000)=a_0+a_1\cos(2\pi)+b_1\sin(2\pi)=a_0+a_1=5\).

Collecting: \(a_0+b_1=4; a_0-a_1=1; a_0-b_1=2; a_0+a_1=5\).

Solving: \(a_0=3; a_1=2; b_1=1. \) [4 equations in 3 unknowns]

Closed-Form Solution to the Linear System of Equations

- General: Signal is sampled at \(S\) sample/seconds; has period \(T\) seconds; bandlimited to \(B\) Hertz.
- Data: \(\{x(1/S),x(2/S),x(3/S),…,x(ST/S)=x(T)\}\) (sampling every 1/S seconds, up to \(T\) seconds).
- Solve: \(ST=N\) equations in \(2BT+1\) unknowns:

\[
\begin{align*}
x(t) &= a_0 + a_1 \cos(2\pi t/T) + \ldots + a_BT \cos(2\pi BT t/T) \\
&\quad + b_1 \sin(2\pi t/T) + \ldots + b_BT \sin(2\pi BT t/T)
\end{align*}
\]

Small Numerical Example, Continued

- We observe these samples:

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</table>
- \(a_0=(1/4)[4+1+2+5]=\text{average value of } x(t)=3\).
- \(a_1=(2/4)[4\cos(\pi/2)+1\cos(\pi)+2\cos(3\pi/2)+5\cos(2\pi)]=2\).
- \(b_1=(2/4)[4\sin(\pi/2)+1\sin(\pi)+2\sin(3\pi/2)+5\sin(2\pi)]=1\).

Answer: \(x(t)=3+2\cos(2\pi t/10000)+\sin(2\pi t/10000)\).

MUCH easier than solving \(ST\) simultaneous linear equations in \(2BT+1\) unknowns if \(ST \& 2BT\) large!
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Using Matlab to Compute Spectra

• Matlab’s “F=fft(X)” computes the following:
  \[ F(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi nk/N} \]
  for \( k = 1, \ldots, N \)

• \( a_k \) = \( k \)th element of “\( 2/N \times \text{real}(\text{fft}(X)) \)”, \( k = 1, 2, \ldots \)
• \( b_k \) = \( k \)th element of “\( -2/N \times \text{imag}(\text{fft}(X)) \)”, \( k = 1, 2, \ldots \)
• \( c_k \) = \( k \)th element of “\( 2/N \times \text{abs}(\text{fft}(X)) \)”, \( k = 1, 2, \ldots \)

• \( X = \text{vector of sampled signal; } N = \text{ST (integer); we need to ensure } S > 2(\text{maximum frequency of } x(t)) \).

Using Matlab’s “fft” to compute spectra

• “fft” output has mirror symmetry (\( e^{-i2\pi(n-1)(k-1)/N} \)).
  \[ \text{DELETE THE 2nd HALF OF ITS OUTPUT!} \]

• Watch indexing: Matlab: 1 to N. Math: 0 to N-1.

• \( (2/N) \times \text{real}(\text{fft(\{x(0)…x(N-1)\},N))) \) gives
• \( [a_0, a_1, a_2, \ldots, a_{N/2}, b_{N/2}, b_{N/2-1}, b_0, b_1, \ldots, b_{N/2-2}] \]

• \( -(2/N) \times \text{imag}(\text{fft(\{x(0)…x(N-1)\},N))) \) gives
• \( [0, b_1, b_2, \ldots, b_{N/2}, 0, -b_{N/2}, -b_{N/2-1}, \ldots, -b_1] \)

• \( (2/N) \times \text{abs}(\text{fft(\{x(0)…x(N-1)\},N))) \) gives
• \( [c_0, c_1, \ldots, c_{N/2}, c_{N/2-1}, c_{N/2-2}, \ldots, c_2, c_1] \)

Small Numerical Example, Continued

<table>
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<th>n=4000t</th>
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</tr>
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<tbody>
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<td>4</td>
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</tr>
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Reorder samples of x(t) to start with x(0)=x(4/4000)=5. Then use:

\( (2/4) \times \text{real}(\text{fft([x(0)…x(N-1)],4))) \) outputs [6 2 0 2] so \( a_0=6/2=3 \) and \( a_1=2 \).

Another Simple Numerical Example

• GIVEN: \( x(t)=\cos(2\pi 440t) \) sampled at 8192 Hertz.
• >>X=cos(2*pi*440*[0:8191]/8192);%T=1 second.
• >>F=(2/8192)*abs(fft(X,8192)) outputs all zeros except one at F(441)=1 and F(7753=8193-440)=1.

• So how do we interpret this Matlab output?
• Remember to delete the mirror image output.
• There is a single sinusoid, but what frequency?

Another Numerical Example, Continued

• S=sampling rate; T=period; N=ST=length of X.
• NOTE: \( x(t) \) is assumed by “fft” to be a single period.
• \( k \)th index is frequency \( (k-1)/T=S(k-1)/N \) Hertz for \( k=1, 2, \ldots N/2 \) (require maximum frequency<\( S/2 \)).
• Frequency resolution (separation) is 1/T Hertz.

• HERE: \( x(t) \) is samples of one sinusoid at frequency \( (441-1)(8192 \text{ sample/sec.(sec.)}/8192 \text{ samples})=440 \text{ Hz.} \)
• Plot only \textbf{first half of } \( F \) to get the spectrum of \( x(t) \)!
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Periodic and Bandlimited Signals

• General: Signal periodic with period=T seconds, & bandlimited to maximum frequency=B Hertz, is completely characterized by 2BT+1 numbers.

• If we know (somehow) that this is bandlimited to 35 Hertz, then 2(35)(0.2)+1=15 numbers suffice. (Every other harmonic was zero in its spectrum.)

Example: Spectrum of periodic signal

Period=0.2 second (5 complete periods occur in 1 second). Other than that, all you can say is that this looks weird. But:

x(t)=15cos(10πt) + 5cos(30πt) + 3cos(50πt) + 2cos(70πt), 0≤t<1.

So the spectrum of this signal is actually quite simple:

Example: Spectrum of periodic signal

Great! But, given this waveform, how do I compute that formula?

Periodic and Bandlimited Signals

• General: Signal periodic with period=T seconds, & bandlimited to maximum frequency=B Hertz, is completely characterized by 2BT+1 numbers.

• If we know (somehow) that this is bandlimited to 35 Hertz, then 2(35)(0.2)+1=15 numbers suffice. (Every other harmonic was zero in its spectrum.)

Example: Spectrum of periodic signal

• The Matlab commands I used to make this example:
  T=0:0.01:99.99; %Sampling rate=1/0.01=100 samples/second.
  Duration=99.99 seconds. Length=100/0.01=10000 samples.
  X=15*cos(10*pi*T)+5*cos(30*pi*T)+3*cos(50*pi*T)+2*cos(70*pi*T);
  %Generates sampled signal. Of course, we don’t “know” this formula.
  We want to compute the spectrum of this signal from its samples X.

  FX=2/10000*abs(fft(X)); F=[0:4999]/100; plot(F,FX(1:5000))
  %Computes the spectrum FX and plots it vs. actual frequency F Hz.

Example: Spectrum of periodic signal

• Sampling rate=100 samples/second>2(max. frequency)=2(35 Hertz).
• Frequency resolution (discretization)=1/duration=1/100=0.01 Hertz.
  The longer the duration T, the higher the frequency resolution 1/T.

• If you want to be lazy, just plot(abs(fft(X))) where X=sampled signal.
• Peak at index K is at frequency f=(K-1)(sampling rate)/(length(X)).
  Here, peak at index K=501 is at frequency (501-1)(100)/10000=5 Hz.
  Peak at index K=1501 is at frequency (1501-1)(100)/10000=15 Hertz.
  Peaks at indices K=2501 and 3501 are at frequencies 25 and 35 Hertz.

Example: Spectrum of periodic signal

• Periodic signals (including musical notes) can be expanded in a Fourier series.
• The signal is a sum of sinusoids at frequencies that are integer multiples of 1/(period of signal).
• If signal is both bandlimited and periodic, then the Fourier series has only a finite #terms, and we can compute coefficients from its samples.
• There is a closed-form solution to the resulting linear system; Matlab can compute it easily.
• Use f=(K-1)S/N to get frequencies from index.

Conclusion