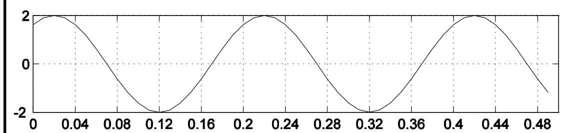
Engin. 100: Music Signal Processing Lab #2: Computing and Visualizing The Frequencies of Musical Tones

- Computing frequency of a sampled sinusoid
- Visualizing and interpreting numerical results using semi-log and log-log plots

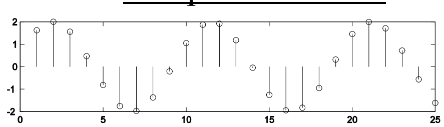
Musical pure tones are sinusoids



Formulae for this:
$x(t)=2\cos(10\pi t - \pi/5)$
$x(t)=2\cos(10\pi(t - 0.02))$

Amplitude=2
Frequency=5 Hertz
Period=0.2 seconds

Sampled Sinusoid



$x(t) = 2\cos(10\pi t - \pi/5)$. Substitute $t=n/50=0.02n$:

$x[n]=2\cos(0.2\pi n-\pi/5)$. Sampling rate=50 Hertz

x[n]	x[0]=1.62	x[1]=2.00	x[2]=1.62	x[3]=0.62
x[n]	$2\cos(0.2\pi 0-\pi/5)$	$2\cos(0.2\pi 1-\pi/5)$	$2\cos(0.2\pi 2-\pi/5)$	$2\cos(0.2\pi 3-\pi/5)$

Reconstruct Sinusoid from Samples

- $x(t)$ is <u>pure sinusoid</u>, sampled at S sample/second:
- $x(t)=A\cos(2\pi ft+\theta) \leftrightarrow x[n]=A\cos(2\pi fn/S+\theta)$.
- <u>Reconstruct</u> $x(t)$ from samples $x[n]=A\cos(Bn+\theta)$: Set $n=St \rightarrow x(t)=A\cos(BSt+\theta)$ where B is given and S=sampling rate in samples/second="Hertz."
- <u>Example</u>: $x[n]=3\cos(0.4\pi n+1)$, S=40 "Hertz."
- Then $x(t)=3\cos(16\pi t+1)$, an 8 Hertz sinusoid.

How can we compute the frequency f of a sinusoid from its samples x[n]?

- <u>GIVEN</u>: Samples x[n] of a pure sinusoid: $x(t)=A\cos(2\pi ft+\theta)$. A,f,θ are all unknown.
- <u>GOAL</u>: Compute the frequency f in Hertz from samples $x[n]=A\cos(2\pi fn/1000+\theta)$.
- <u>Given</u>: sampling rate=1000 samples/second. Replace 1000 with actual rate in following.
- Don't care about A and θ; <u>can</u> compute them.

How can we compute the frequency f of a sinusoid from its samples x[n]?

- "All you need is trig…"
- Recall (?) the cosine addition formulae:
- $\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$;
- $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$.
- Add and subtract to get this identity:
- $\boxed{\cos(x+y)+\cos(x-y)=2\cos(x)\cos(y)}$
- We will use this identity over and over!

How to Tune a Piano:

- Let $x=2\pi 441t$ and $y=2\pi t$ in above trig formula.
- $\cos(2\pi 442t)+\cos(2\pi 440t)=2\cos(2\pi t)\cos(2\pi 441t)$.
- The sum of two sinusoids at close frequencies is a sinusoid at the average of the frequencies with a (sinusoidally) *time-varying* amplitude!
- Signal gets louder and softer with period=0.5 sec.
- The slower the period, the closer the 2 frequencies.
- "You can tune a piano, but you can't tuna fish"

Computing frequency of a sinusoidal signal sampled at 1000 samples/second

- Let $x=2\pi fn/1000+\theta$ and $y=2\pi f1/1000$ in the trig identity:

$$\cos(2\pi f(n+1)/1000+\theta)+\cos(2\pi f(n-1)/1000+\theta)=2\cos(2\pi f1/1000)\cos(2\pi fn/1000+\theta).$$

- Sampled sinusoid is $x[n]=x(t=n/1000)=\cos(2\pi fn/1000+\theta)$.
- Substituting gives $x[n+1]+x[n-1]=2\cos(2\pi f/1000)x[n]$.
- Solving: $\boxed{f=(1000/2\pi)\arccos\{(x[n+1]+x[n-1])/2x[n]\}.}$
- Use this formula in Lab to compute frequency from x[n].

Example: Use of this Formula

- PROBLEM:
- A sinusoid is sampled at 1500 samples/second.
- All but 3 of the samples, and times, are garbled:
- x[n]={…*,*,*,3,7,4,*,*,*…}; *=garbled value.
- "What's the frequency, Kenneth"?
- SOLUTION:

$f=[1500/(2\pi)]\arccos[(x[n+1]+x[n-1])/(2x[n])]$. Plug:

$[1500/(2\pi)]\arccos[(3+4)/(2\cdot 7)]=1500/(2\pi)(\pi/3)=250$ Hertz.

(Dis)Advantages of this Formula

- Advantages in using this formula:
- Very simple to implement-use simple DSP chip.
- Fast tracking of sudden frequency changes.
- Can use outliers (weird values) to segment signal.

- Disadvantages in using this formula:
- Very sensitive to additive noise in the data x[n].
- What if x[n]=0 for some n? Must divide by 0!
- Arc-cosine function is sensitive to small changes.

Engin. 100: Music Signal Processing Lab #2: Computing and Visualizing The Frequencies of Musical Tones

- Computing frequency of a sampled sinusoid
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Data Visualization

- GIVEN: A set of numbers (data) vs. another set.
- GOAL: Find a formula for the set of numbers.

- EXAMPLE: Three sets of data vs. t=1,2,3,4,5:
- SET A: {48, 72, 108, 162, 243}
- SET B: {32, 128, 288, 512, 880}
- SET C: {8, 64, 512, 1000} (one time is missing!)

Semi-Log Data Plots

- Let {x(1),x(2),x(3)…} be the data vs. time.
- If data obeys formula of form $x(t)=ba^t$:
- <u>log(x)=(t)log(a)+log(b)</u> (use any log base).

- Plotting log(x) vs. t yields a straight line.
- Slope=log(a) and y-intercept=log(b).

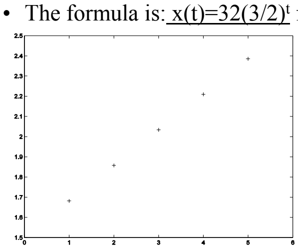
Log-Log Data Plots

- Let {x(1),x(2),x(3)…} be the data vs. time.
- If data obeys formula of form $x(t)=bt^n$:
- <u>log(x)=(n)log(t)+log(b)</u> (use any log base).

- Plotting log(x) vs. log(t) yields a straight line.
- Slope=n and y-intercept=log(b).

Example: Data Set A Above

- Recall x(t)={48, 72, 108, 162, 243} for t=1,2,3,4,5.
- Slope of line below=(2.4-1.7)/(5-1)=0.175=$\log_{10}(3/2)$.
- Y-intercept of line below=1.7-0.175=1.505=$\log_{10}(32)$.
- The formula is: <u>$x(t)=32(3/2)^t$</u> for t=1,2,3,4,5.

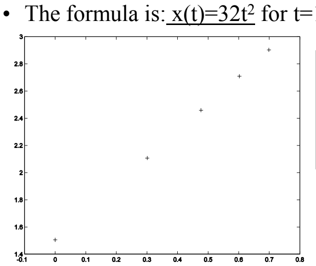


<u>Matlab commands for plot</u>:

```
X=[48 72 108 162 243];
plot([1:5],log10(X),'+')
axis([0 6 1.5 2.5]) %so no
%points plotted on the axes
```

Example: Data Set B Above

- Recall x(t)={32, 128, 288, 512, 800} for t=1,2,3,4,5.
- <u>Slope</u> of line below=(2.9-1.5)/(0.7-0)=2=n (exponent).
- <u>Y-intercept</u> of line below=1.5=$\log_{10}(32)=\log_{10}(b)$. b=32.
- The formula is: <u>$x(t)=32t^2$</u> for t=1,2,3,4,5.

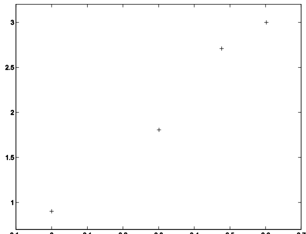


<u>Matlab commands for plot</u>:

```
X=[32 128 288 512 800];
plot(log10([1:5]),log10(X),'+')
axis([-.1 .8 1.4 3]) %so no
%points plotted on the axes
```

Example: Data Set C Above

- x(t)={8, 64, 512, 1000} for *some four* of t=1,2,3,4,5.
- <u>From log-log plot</u>: Break in line means *t=3 is missing*.
- <u>NEXT SLIDE</u>: log-log plot {8,64,512,1000} vs. {1,2,4,5}

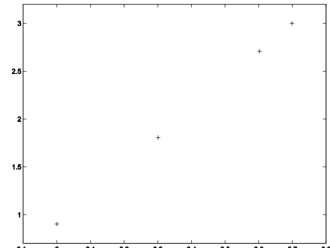


<u>Matlab commands for plot</u>:

```
X=[8 64 512 1000];
plot(log10([1:4]),log10(X),'+')
axis([-.1 .7 .7 3.2]) %so no
%points plotted on the axes
```

EXAMPLE: DATA SET C ABOVE

The <u>missing value</u> at t=3 is interpolated to about 300 [288].
<u>Slope</u> of line below=(3-0.9)/(0.7-0)=3=n=exponent.
<u>Y-intercept</u> of line below=0.9=$\log_{10}(8)=\log_{10}(b)$. b=8.
The formula is: <u>$x(t)=8t^3$</u> for t=1,2,3,4,5.



<u>Matlab commands for plot</u>:

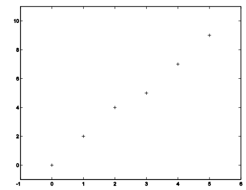
```
X=[8 64 512 1000]; %Data.
T=[1 2 4 5]; %Now know T.
plot(log10(T),log10(X),'+')
axis([-.1 .8 .7 3.2]) %so no
%points plotted on the axes.
```

What will you do in Lab #2?

- Download a sampled signal from Ctools site: A tonal version of the chorus of "The Victors."
- Load into Matlab; segment (chop up) into notes.
- Apply formula to compute frequency of each note.
- Make log-log and semi-log plots of frequencies.
- Discern the formula relating frequencies of notes.
- NOTE: "Accidentals" are all missing; but you can infer their existence & frequencies from your plot!

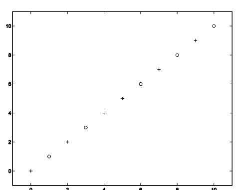
BUT: In Lab #2 there will be many missing frequencies, since they do not appear in "The Victors." These are *accidental* notes. You must handle this differently, as in the following example:

Example: We are given the values x(t)={1, 4, 16, 32, 128, 512}. What are, and and at what times are, the missing values of x(t)? Semi-log plot of x(t) is shown below. How to interpret it? Looks like the previous example, but the slope of the break is ½, not 2!



```
X=[1 4 16 32 128 512];T=[0:5];
plot(T,log2(X),'+') %Semi-Log
axis([-1 6 –1 11]) %for nice plot
```

NOW: there are missing values everywhere *except* in the middle! The plot with all of the missing values included is shown below. This is what happens in Lab #2 with frequencies from "The Victors."



```
X=[1 4 16 32 128 512];%as before
T=[0 2 4 5 7 9]; %actual T values
plot(T,log2(X),'+'),hold
U=[1 3 6 8 10]; %missing T values
plot(U,U,'o')    %missing X values
%since slope=1 just plot U vs. U.
axis([-1 11 –1 11]) %for nicer plot
```

The missing X values have log2(X)=U=[1 3 6 8 10] (slope=1). The missing X values are X=[2 8 64 256 1024]! So $x(t)=2^t$ here.

Dimensional Analysis Example #1

- GOAL: Determine formula for the period of a swinging pendulum, <u>without any physics</u>!
- MODEL: Period=$(\text{mass})^a(\text{length})^b g^c$ where g=acceleration of gravity (32 ft/sec^2) and a,b,c are unknown constants to be found
- SOLUTION: Equate exponents on both sides of the formula using dimensional analysis

Dimensional Analysis Example #1

- Period=$(\text{mass})^a(\text{length})^b g^c$. Dimensions:
- time=$(\text{mass})^a(\text{length})^b\,(\text{length/time}^2)^c$
- Mass: a=0. Length: 0=b+c. Time: 1=-2c.
- Solve: a=0, b=1/2, c=-1/2
- Formula: Period=$[\text{Length}/g]^{½}$
- Actually: Period=$2\pi[\text{Length}/g]^{½}$
- 2π dimensionless-can't infer it dimensionally

Dimensional Analysis Example #2

<u>Allow me to ask you 3 questions:</u>

If 1.5 people can build 1.5 cars in 1.5 days, how many cars can 9 people build in 9 days?

Do problems like this give you a headache?

Would you like to be able to solve problems like this without having to do any thinking?

Dimensional Analysis Example #2

If 1.5 people can build 1.5 cars in 1.5 days,
how many cars can 9 people build in 9 days?

Look at the dimensions (units) of the given quantities:
(1.5 cars)/(1.5 days)/(1.5 people)=2/3 cars/day/people.

(2/3 cars/day/people)(9 days)(9 people)=54 cars.
No thinking required-just use dimensions (units).

Conclusion

- Sampling: computer can compute frequency of a sinusoid from 3 consecutive samples.
- Semi-log plot of $x=ba^t$ is a straight line.
- Log-log plot of $x=bt^n$ is a straight line.
- Dimensional analysis can give you answer, even if you have no idea what's going on!