# Engin. 100: Music Signal Processing Lab #3: Signal Spectra and Filtering

- How do we define the spectrum of a signal?
- Why can we sample without losing information?
- How do we compute the spectrum of a signal?
- How do we use Matlab to compute the spectrum?
- Why should we bother to compute the spectrum?

## What is the spectrum of a signal?

- **In Engin. 100, we will only define spectra of periodic (repeating) signals. How come?**

1. Each musical note played by an instrument is a periodic signal, so this is what we care about.
2. Definition and computation are much easier.
3. Actually, a non-periodic signal can be viewed as a periodic signal with a VERY long period.

## Fourier Series of Periodic Signals

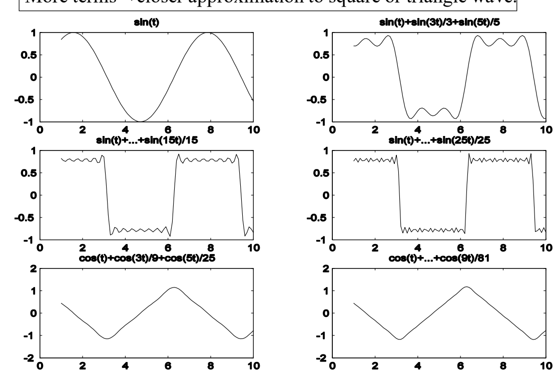
- Any real-world periodic signal with period=T (so x(t)=x(t+T) for all t) can be expanded as:

- $x(t)=c_0+c_1\cos(2\pi t/T-\theta_1)+c_2\cos(4\pi t/T-\theta_2)+\ldots$

| DC constant | fundamental period=T | harmonic period=T/2 |
|---|---|---|

The **spectrum** of x(t) is just a stem plot of the amplitudes $\{c_k\}$ vs. the frequencies $\{k/T\}$ Hertz.

Convergence of Fourier series for square and triangle waves.
More terms→closer approximation to square or triangle wave.




## Fourier Series of Musical Signals

- Phase shift $\theta$ cannot be perceived by the ear.
- No DC term in musical signals. So can use:

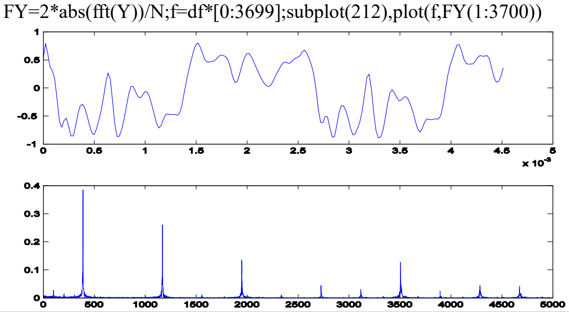
$x(t)=a_1\cos(2\pi t/T)+a_2\cos(4\pi t/T)+a_3\cos(6\pi t/T)+\ldots$

fundamental harmonic harmonic

- This model originally proposed by Bernoulli.
- Coefficients $a_n$→timbre (TAM-ber) of sound:
- Is why a clarinet sounds different from a flute.
- Plot of $a_k$ vs. frequency k/T Hertz: sound spectrum.

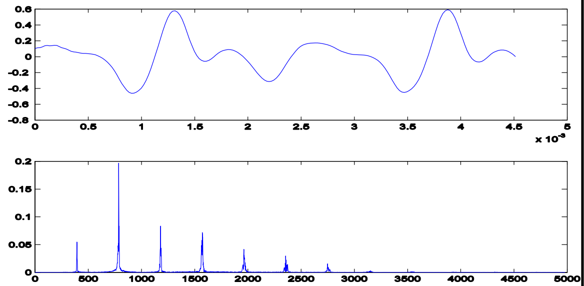
## Clarinet Time Waveform and Spectrum Note "G"(392 Hz) Sampled 44100 Hz

```
clear; [X S]=audioread('proj3.wav');N=32768;XX=reshape(X,N,13,4);
Y=XX(:,1,2);dt=1/S;df=S/N;t=dt*[0:199];subplot(211),plot(t,Y(1:200))
FY=2*abs(fft(Y))/N;f=df*[0:3699];subplot(212),plot(f,FY(1:3700))
```



## Trumpet Time Waveform and Spectrum Note “G”(392 Hz) Sampled 44100 Hz

```
clear; [X S]=audioread('proj3.wav');N=32768;XX=reshape(X,N,13,4);
Y=XX(:,1,3);dt=1/S;df=S/N;t=dt*[0:199];subplot(211),plot(t,Y(1:200))
FY=2*abs(fft(Y))/N;f=df*[0:3699];subplot(212),plot(f,FY(1:3700))
```

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## Fourier Series of Periodic Signals

- Any real-world periodic signal with period=T can be expanded in either of these two forms:
- $x(t)=c_0+c_1\cos(2\pi t/T-\theta_1)+c_2\cos(4\pi t/T-\theta_2)+\ldots$
- $x(t)=a_0+a_1\cos(2\pi t/T)+a_2\cos(4\pi t/T)+\ldots$
- $+b_1\sin(2\pi t/T)+b_2\sin(4\pi t/T)+\ldots$
- Coefficients in these two forms are related by:
- $A\cos(t)+B\sin(t)=C\cos(t-\theta)$ (cf. Lab #1) where $C=(A^2+B^2)^{1/2}$; $\tan\theta=B/A$; $A=C\cos\theta$; $B=C\sin\theta$.

## Example: Period=0.001 seconds

- x(t) has period=T=0.001 sec. x(t) has expansion:
- $x(t)=a_0+a_1\cos(2\pi 1000t)+a_2\cos(2\pi 2000t)+\ldots$
- $+b_1\sin(2\pi 1000t)+b_2\sin(2\pi 2000t)+\ldots$
- 0 1000 Hz (fund.) 2000 Hz (harmonic)
- In general, this expansion is infinite and requires infinitely high frequencies at integer multiples of the fundamental frequency=1/period=1000 Hz.
- But what if the signal is bandlimited to 8000 Hz?
- Means: It has no sinusoids with freqs>8000 Hz.

## Example: Period=0.001 seconds; Bandlimited to 8000 Hertz

- $x(t)=a_0+a_1\cos(2\pi 1000t)+\ldots+a_8\cos(2\pi 8000t)$
- $+b_1\sin(2\pi 1000t)+\ldots+b_8\sin(2\pi 8000t)$
- Now the expansion has only finite length. In fact:
- If x(t) is periodic with period=0.001 seconds and bandlimited (maximum frequency) to 8000 Hertz:
- Then it’s completely characterized by 17 numbers: $\{a_0,\ a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,\ b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8\}$.

## Sampling Signals with Period=T seconds and Bandlimited to B Hertz

- IDEA: If a signal is completely characterized by 2BT+1 numbers, we can compute those 2BT+1 numbers by sampling 2BT+1 times per period!
- THAT IS: Sample *faster* than 2BT times/period.
- Sampling rate>(2BT times)/(T second)=2B Hertz. So need: sampling rate>2(maximum frequency).

## Sampling Signals with Period=T seconds and Bandlimited to B Hertz

- Sampling rate>(2BT times)/(T second)=2B Hertz. Need: sampling rate>2(maximum frequency).
- Note: Period=T does not matter to sampling rate! We can make T arbitrarily large; this still holds.
- Sampling theorem: co-discovered Claude Shannon, U-M CoE Class of 1936 (EE). Bust outside EECS.

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## Closed-Form Solution to the Linear System of Equations

- General: Signal is sampled at S sample/seconds; has period=T seconds; bandlimited to B Hertz.
- Data: {x(1/S),x(2/S),x(3/S),…x(ST/S)=x(T)} (sampling every 1/S seconds, up to T seconds).
- Solve: ST=N equations in 2BT+1 unknowns:
- Set t={1/S, 2/S,…,(ST)/S=T} in the equation:
- $x(t)=a_0+a_1\cos(2\pi t/T)+\ldots+a_{BT}\cos(2\pi BTt/T)$
- $+b_1\sin(2\pi t/T)+\ldots+b_{BT}\sin(2\pi BTt/T)$

## Closed-Form Solution to the Linear System of Equations

- There is a closed-form solution (formula):
- $a_k=(2/N)\sum x(n/S)\cos(2\pi nk/N)$ where N=ST.
- $b_k=(2/N)\sum x(n/S)\sin(2\pi nk/N)$ where N=ST.
- $a_0=(1/N)\sum x(n/S)$=average value of x(n/S).
- Computed for k=0 to BT (2BT+1 numbers).
- All sums are taken from n=1 to n=N=ST.
- Where does this formula come from? See the Appendix of Lab #3 (MUCH algebra!).

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## Using Matlab to Compute Spectra

- Matlab's "F=fft(X)" computes the following:
- $\text{fft}(X)=\sum x(n-1)e^{-i2\pi(n-1)(k-1)/N}$ for k=1,…N where sum is from n=1 to N, and $e^{-ix}=\cos(x)-i\cdot\sin(x)$.
- $a_{k-1}$=$k^{th}$ element of "2/N*real(fft(X))", k=1,2…
- $b_{k-1}$=$k^{th}$ element of "-2/N*imag(fft(X))",k=1,2…
- $c_{k-1}$=$k^{th}$ element of "2/N*abs(fft(X))", k=1,2…
- X=vector of sampled signal; N=ST (integer); we need to ensure S>2(maximum frequency of x(t)).
- T=period in seconds=(#samples per period)/S.

## Using Matlab's "fft" to compute spectra

- "fft" output has mirror symmetry ($e^{-i2\pi(n-1)(k-1)/N}$).
- **DELETE THE 2nd HALF OF ITS OUTPUT!**
- Watch indexing: Matlab: 1 to N. Math: 0 to N-1.
- (2/N)*real(fft([x(0)…x(N-1)],N)) gives
- $[2a_0\ a_1\ a_2 \ldots a_{N/2-2}\ a_{N/2-1}\ a_{N/2}\ a_{N/2-1}\ a_{N/2-2} \ldots a_2\ a_1]$
- -(2/N)*imag(fft([x(0)…x(N-1)],N)) gives
- $[0\ b_1\ b_2 \ldots b_{N/2-2}\ b_{N/2-1}\ 0\ -b_{N/2-1}\ -b_{N/2-2} \ldots -b_2\ -b_1]$
- (2/N)*abs(fft([x(0)…x(N-1)],N) gives
- $[2c_0\ c_1\ c_2 \ldots c_{N/2-2}\ c_{N/2-1}\ c_{N/2}\ c_{N/2-1}\ c_{N/2-2} \ldots c_2\ c_1]$

## Small Illustrative Numerical Example

- GIVEN: $x(t)=\cos(2\pi 440t)$ sampled at 8192 Hertz.
- >>X=cos(2*pi*440*[0:8191]/8192);%T=1 second.
- >>F=(2/8192)*abs(fft(X,8192)) outputs all zeros except one at F(441)=1 and F(7753=8193-440)=1.
- So how do we interpret this Matlab output?
- Remember to delete the mirror image output.
- There is a single sinusoid, but what frequency?

## Small Illustrative Numerical Example

- S=sampling rate; T=period; N=ST=length of X.
- NOTE: X is assumed by "fft" to be a single period.
- $k^{th}$ index is frequency (k-1)/T=S(k-1)/N Hertz for k=1,2…N/2 (require maximum frequency<S/2).
- Frequency resolution (separation) is 1/T Hertz.
- HERE: X is samples of one sinusoid at frequency (441-1)(8192 sample/sec.)/(8192 samples)=440 Hz.
- Plot only **first half of F** to get the spectrum of X!

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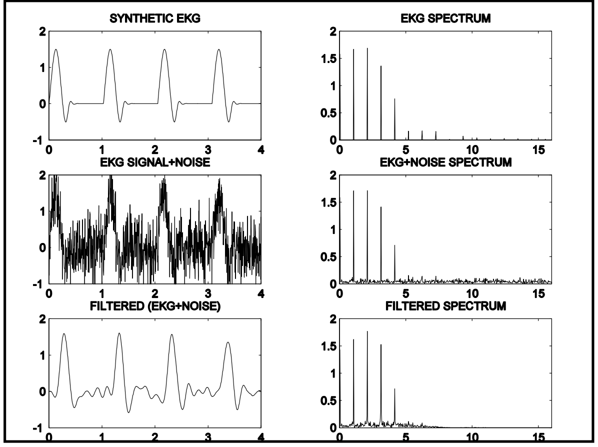
## Spectral filtering of noisy signals

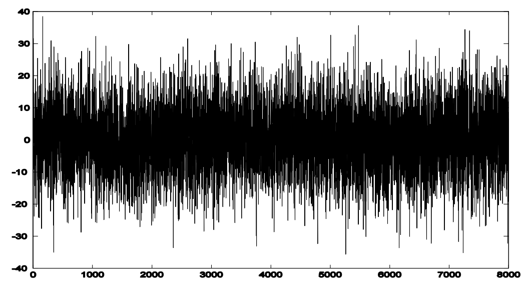
- **GIVEN**: Observe: y=x+n=desired signal+noise.
- **GOAL**: Filter out as much noise n as possible.
- **KNOW**: Signal bandlimited to f=(K-1)S/N Hertz.
- **IDEA**: Eliminate frequencies above F Hertz, since they must be noise. Do this using "fft" as follows:
- y=x+n; fy=fft(y); fy(K:N+2-K)=0; z=real(ifft(fy));
- z is *low-pass filtered*, hence most noise eliminated.
- **BUT**: Can't eliminate noise below F Hertz, since those noise components overlap signal components.

## Low-pass filtering a noisy EKG signal

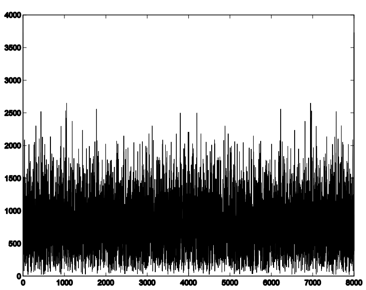
- EKG=Electrocardiogram (heart) electric signal.
- Periodic (we hope!) with period about 1 second (note 60 beats per minute=1 beat per second).
- Components at 1,2,3…Hertz up to about 7 Hertz.
- **So**: We eliminate all frequencies above 7 Hertz.
- **Result**: Most noise gone; not low frequencies.
- See next slide for signals and their spectra.




## Noisy Signal (from First Lecture) What is the signal buried in noise? How do we recover the signal?

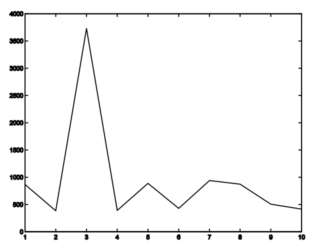


## Spectrum of noisy signal: abs(fft(X)) Huh? But look at the vertical scale



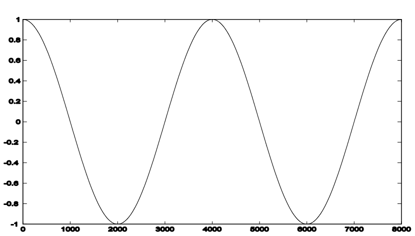
Spectrum also looks random. Look carefully: there is a large component very close to zero, at the extreme left. Zoom in to this (see next slide).

## Spectrum of noisy signal: Zoom in. Now it's clear what the signal is!



There is a large component (peak) at index K=3. This is a sinusoid at the frequency f=(K-1)S/N= (3-1)8000/8000= 2 Hertz (is right). Next, we filter it.

## Filtering Noisy Signal: Set all of the components to zero except K=3,7999



Matlab code used:

```
T=[0:7999]/8000;
X=cos(4*pi*T);
N=randn(1,8000);
Y=X+10*N;
F=fft(Y);
F(4:7998)=0;
F([1 2 8000])=0;
Z=real(ifft(F));
```

NOTE: Need to keep both K=3 and its mirror index K=8000+2-3. Mirror index of K is N+2-K due to Matlab indexing starting at 1. If counting from k=0 to N-1, so f=kS/N, the mirror index of k is N-k.

## Conclusion

- Periodic signals (including musical notes) can be expanded in a Fourier series.
- The signal is a sum of sinusoids at frequencies that are integer multiples of 1/(period of signal).
- If signal is both bandlimited and periodic, then the Fourier series has only a finite #terms, and we can compute coefficients from its samples.
- There is a closed-form solution to the resulting linear system; Matlab can compute it easily.
- Use f=(K-1)S/N to get frequencies from index.