EECS 564 VECTOR GAUSSIAN DETECTION Winter 1999

GIVEN: Observation R of a Gaussian random *vector* r, where:

PDFs: $p_{r|H_1}(R|H_1) \sim N(m_1, K_1)$ where m_0, m_1 are mean vectors
$p_{r|H_0}(R|H_0) \sim N(m_0, K_0)$ K_0, K_1 covariance matrices.

DEF: $|K_0| = det(K_0)$; $Q_0 = K_0^{-1}$; $\Delta m = m_1 - m_0$.

LRT: $$\Lambda(R) = \frac{p_{r|H_1}(R|H_1)}{p_{r|H_0}(R|H_0)} = \frac{\frac{1}{(2\pi)^{N/2}\sqrt{|K_1|}} e^{-(R-m_1)^T K_1^{-1}(R-m_1)/2}}{\frac{1}{(2\pi)^{N/2}\sqrt{|K_0|}} e^{-(R-m_0)^T K_0^{-1}(R-m_0)/2}}$$

log LRT: $\ell(R) = \frac{1}{2}[(R-m_0)^T Q_0 (R-m_0) - (R-m_1)^T Q_1 (R-m_1)]$
(subtract $\frac{1}{2}(\log|K_0| - \log|K_1|)$ from threshold $\log\eta$).

Consider two cases: $m_0 \neq m_1$ and $K_0 \neq K_1$:

$m_o \neq m_1$: $\ell(R) = \frac{1}{2}[(R-m_0)^T Q_0 (R-m_0) - (R-m_1)^T Q_0 (R-m_1)]$
$K_0 = K_1$ $\ell(R) = (\Delta m)^T Q_0 R + \frac{1}{2}[m_0^T Q_0 m_0 - m_1^T Q_0 m_1]$ (constant).

LRT: $\ell(R) = (\Delta m)^T Q_0 R \gtrless \gamma \rightarrow$ **correlation receiver**.
$\ell(R) = \sum_{i=1}^N ((\Delta m)^T Q_0)_i R_i \rightarrow$ **matched filter**.

N-P: Under H_i, $\ell(r) \sim N((\Delta m)^T Q_0 m_i, (\Delta m)^T Q_0 \Delta m)$.
TEST $\rightarrow$use previous results for *scalar* Gaussian detection:
scalar: $P_F = erfc\left(\frac{\gamma - m_0}{\sigma}\right)$; $P_D = erfc\left(\frac{\gamma - m_1}{\sigma}\right) = erfc\left(\frac{\gamma - m_0}{\sigma} - d\right)$.
Fisher discriminant$= d^2 = \frac{(E[\ell|H_1] - E[\ell|H_0])^2}{\sigma_\ell^2} = (\Delta m)^T Q_0 \Delta m$.

$K_0 \neq K_1$: $\ell(R) = \frac{1}{2}(R-m_0)^T (Q_0 - Q_1)(R-m_0)$. $\Delta Q = Q_0 - Q_1$.
$m_0 = m_1$ WLOG $m_0 = 0 \rightarrow \ell(R) = \frac{1}{2} R^T (\Delta Q) R$ **quadratic form**.

scalar: Distinguish Gaussians with different variances$\rightarrow R^2 \gtrless \gamma$.
vector: $\Delta Q > 0$ $\rightarrow$inside/outside ellipsoid: $R^T (\Delta Q) R \gtrless \gamma$.

signal: $K_0 = \sigma^2 I$ (noise); $K_1 = \sigma^2 I + K_s$ (random signal+noise):
+noise $\rightarrow \sum_{i=1}^N \left(\frac{1}{\sigma^2} - \frac{1}{\sigma^2 + \lambda_i}\right)(R_i')^2 \gtrless \gamma$ ellipsoid since $\frac{1}{\sigma^2} > \frac{1}{\sigma^2 + \lambda_i}$.
$R' = [\phi_1 \cdots \phi_N]^T R$: matrix of eigenvectors decorrelates.

GIVEN: Gaussian detection with $m_0 \neq m_1$, $K_0 = K_1 = K$.
GOAL: Choose m_0, m_1 to maximize $d^2 = (\Delta m)^T Q_0 \Delta m$
subject to constraints $||m_0||^2, ||m_1||^2 \leq E^2$=energy.

$K = \sigma^2 I$: max $||\Delta m||^2 \rightarrow m_1 = -m_0$ for any m_0 where $||m_0|| = E$.
$m_1 = -m_0$: opposite sides of circle/sphere: *antipodes.*

K diag: Let $K = diag[\sigma_i^2]$. Clearly still want $m_1 = -m_0$.
Different noise strengths in N independent channels.
max $d^2 = \sum_{i=1}^N \frac{(\Delta m)_i^2}{\sigma_i^2} = \sum_{i=1}^N \frac{4(m_0)_i^2}{\sigma_i^2}$. Let $\sigma_k^2 < \sigma_i^2, i \neq k$.

SOLN: $(m_0)_k = -E$; $(m_1)_k = E$; $(m_0)_i = (m_1)_i = 0, i \neq k$:
Put *all* signal energy into least-noisy channel.

PROOF: Scale i^{th} component by σ_i: Let $(\tilde{m}_0)_i = \frac{(m_0)_i}{\sigma_i}$.
Problem becomes: max $d^2 = 4\sum_{i=1}^N (\tilde{m}_0)_i^2 = 4||\tilde{m}_0||^2$
subject to constraint $\sum_{i=1}^N \sigma_i^2 (\tilde{m}_0)_i^2 = E^2$ (ellipsoid).
Which points on this ellipsoid are farthest apart?
Points along the k-axis, where σ_k^2 is smallest σ_i^2. QED.

$K \neq I$: Let K have eigenvalues/vectors λ_i, ϕ_i. Let $\lambda_k < \lambda_i, i \neq k$.

SOLN: $m_0 = -E\phi_k$; $m_1 = E\phi_k$: (above: $\phi_k = [0 \ldots 1 \ldots 0]^T$.)
Put *all* signal energy into least-noisy eigenchannel.

PROOF: Let $R' = [\phi_1 \cdots \phi_N]^T R$ and $\Delta m' = [\phi_1 \cdots \phi_N]^T \Delta m$.
$d^2 = (\Delta m)^T K^{-1} \Delta m = (\Delta m')^T diag[\frac{1}{\lambda_i}] \Delta m'$
$d^2 = \sum_{i=1}^N \frac{(\Delta m')_i^2}{\lambda_i}$; same problem as above ($\sigma_i^2 \rightarrow \lambda_i$).
$m_1 = [\phi_1 \ldots \phi_N] m_1' = [\phi_1 \ldots \phi_N][0 \ldots E \ldots 0]^T = E\phi_k$.
$\ell(R) = (\Delta m') diag[\frac{1}{\lambda_i}] R' = \frac{2E}{\lambda_k} R_k' = \frac{2E}{\lambda_k} \phi_k^T R$
$\ell(R) \sim R_k' = \phi_k^T R$ is matched filter: project R onto ϕ_k.