GIVEN: Observe $r(t) = s(t,a) + w(t), \quad 0 \le t \le T$;
$s(t,a)$ is a known function of unknown parameter a.
$w(t)$ WGN (White Gaussian Noise); 0-mean, $S_w(\omega) = \sigma^2$.

GOAL: Compute $\hat{a}_{MAP}(R(t))$; given *a priori* pdf $p_a(A)$.

K-L: Let $\{\phi_i(t)\}$ be *any* complete orthonormal basis.
$r_i = \int_0^T r(t)\phi_i(t)dt; \quad s_i(a) = \int_0^T s(t,a)\phi_i(t)dt$. Project:
$r_i = s_i(a) + w_i, i = 1, 2 \ldots; \quad r = [r_1, r_2 \ldots]^T$.

$p_{r|a}(R|A) = \prod \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(R_i - s_i(A))^2/2\sigma^2} \to \prod e^{(2R_i s_i(A) - s_i^2(A))/2\sigma^2}$

neglecting terms independent of A (don't affect argmax).
Using Parseval's theorem $\int x(t)y(t)dt = \sum x_i y_i$ and
$\log p_{a|r}(A|R) = \log p_{r|a}(R|A) + \log p_a(A) - \log p_r(R) \to$

$$\hat{a}_{MAP} = \underset{A}{\text{argmax}} \left[\left(2\int_0^T R(t)s(t,A)dt - \int_0^T s(t,a)^2 dt \right) / 2\sigma^2 + \log p_a(A) \right]$$

$$\hat{a}_{MLE} = \underset{A}{\text{argmax}} \left[2\int_0^T R(t)s(t,A)dt - \int_0^T s(t,a)^2 dt \right]$$

EX: $r(t) = as(t) + w(t); \quad a \sim N(0, \sigma_a^2)$ (linear, Gaussian prior):

$$\hat{a}_{MLE} = \underset{A}{\text{argmax}} \left[2\int_0^T R(t)As(t)dt - \int_0^T A^2 s^2(t)dt \right] = \frac{\int_0^T R(t)s(t)dt}{\int_0^T s^2(t)dt}.$$

Matched filter again; this time for estimation.

$$\hat{a}_{MAP} = \underset{A}{\text{argmax}} \left[\left(2\int_0^T R(t)As(t)dt - \int_0^T A^2 s^2(t)dt \right) / 2\sigma^2 - \frac{A^2}{2\sigma_a^2} \right]$$

$$\frac{\partial}{\partial A} = 0 \to \hat{a}_{MAP} = \left(\int_0^T R(t)s(t)dt\right) / \left(\int_0^T s^2(t)dt + \frac{\sigma^2}{\sigma_a^2}\right).$$

Gaussian$\to$ $\hat{a}_{MAP}$ efficient$\to$ $\hat{a}_{MAP} = \hat{a}_{LS}$:

$\hat{a}_{LS} = E[a] + \frac{E[a\ell]}{\sigma_\ell^2}(\ell - E[\ell])$=above expression; $\ell = \int r(t)s(t)dt$.

using $E[a\ell] = E[a \int (as^2(t) + w(t)s(t))dt] = \sigma_a^2 \int s^2(t)dt$

and $\sigma_\ell^2 = E[\int\int (as^2(t) + w(t)s(t))(as^2(u) + w(u)s(u))dt\, du$
$= \sigma_a^2 (\int s^2(t)dt)^2 + \sigma^2 \int s^2(t)dt$. Plug these in above.

NOTE: $\lim_{\sigma_a^2 \to \infty} \hat{a}_{MAP} = \hat{a}_{MLE}$ (no a priori); $\lim_{\sigma_a^2 \to 0} \hat{a}_{MAP} = 0$ (why?).

C-R: $\hat{a}$=any estimator for problem formulated overleaf.

Then $E[(\hat{a}-a)^2] \geq \sigma^2/E[\int_0^T (\frac{\partial s(t,A)}{\partial A})^2 dt - \frac{\partial^2}{\partial A^2}\log p_a(A)]$.

For parameter A (non-Bayesian problem), $\hat{a}$ unbiased,
$E[(\hat{a}-A)^2] \geq \sigma^2/\int_0^T (\frac{\partial s(t,A)}{\partial A})^2 dt$ (note no $E[\cdot]$ here).

For linear $s(t,a)=as(t)$, bound=$\sigma^2/\int_0^T s^2(t)dt$=1/SNR.

PROOF: $\frac{\partial}{\partial A}\log p_{r|a}(R|A) = -\frac{\partial}{\partial A}\sum(R_i - s_i(A))^2/2\sigma^2$
$= \frac{1}{\sigma^2}\sum(R_i - s_i(A))\frac{\partial s_i(A)}{\partial A} = \frac{1}{\sigma^2}\sum w_i \frac{\partial s_i(A)}{\partial A}$.

$$I_r(A) = E[(\frac{\partial}{\partial A}\log p_{r|a}(R|A))^2] = \frac{1}{\sigma^4}\sum(\frac{\partial s_i(A)}{\partial A})^2 E[w_i^2]$$
$$= \frac{1}{\sigma^2}\sum(\frac{\partial s_i(A)}{\partial A})^2 = \frac{1}{\sigma^2}\int_0^T (\frac{\partial s(t,A)}{\partial A})^2 dt. \text{ QED.}$$

$$I = -E[\frac{\partial^2}{\partial A^2}\log p_{r,a}(R,A)] = E[I_r(A)] - E[\frac{\partial^2}{\partial A^2}\log p_a(A)].$$

Time Delay Estimation

GIVEN: Observe $r(t)=s(t-A)+w(t), 0\leq t\leq T, w(t)$ WGN.

GOAL: Compute $\hat{a}_{MLE}$. Applications: radar/sonar ranging.

SOLN: $\hat{a}_{MLE} = \underset{A}{\text{argmax}}\left[\int_0^T R(t)s(t-A)dt - \frac{1}{2}\int_0^T s(t-A)^2 dt\right]$

- Usually assume signal energy independent of delay.
- Cross-correlate and look for peak; neglect second term.

C-R: $E[(\hat{a}-A)^2] \geq \sigma^2/\int(\frac{\partial s(t-A)}{\partial A})^2 dt = 2\pi\sigma^2/\int_{-\infty}^{\infty}\omega^2|\hat{S}(\omega)|^2 d\omega$.

Interpretation: Depends on *mean-square bandwidth.*

- $|\hat{S}(\omega)|^2$=density function $\rightarrow \int_{-\infty}^{\infty}\omega^2|\hat{S}(\omega)|^2 d\omega$=mean square.
- Higher frequency components of $s(t)$ most useful.
- This makes sense: trying to localize time for delay.