PROBLEM: \textbf{Observe} \( r(t) = (x + h) + w(t) \text{ under } H_1, \) \( J_0 \triangledown x(t) = E, \) \text{ for } \sigma_0 \gg \bar{\varepsilon}, \text{ with very strong } H_1.

\textbf{Guess:} \( J_0 \triangledown x(t) = E \).

\textbf{Basis Functions:} \( \varphi(t), \phi(t) = \) \( \begin{cases} \frac{\sigma_0}{\sqrt{2}} \sin(\pi t), & \text{if } a < b \leq \pi, \\ \frac{\sigma_0}{\sqrt{2}} \cos(\pi t), & \text{otherwise}. \end{cases} \)

Now, \( J_0 \triangledown x(t) = \int \varphi \triangledown x(t) \, dt = \int \varphi \, \phi \, dt = E \).

\textbf{Derivative of Basis Functions:} \( \varphi(t) = \frac{\sigma_0}{\sqrt{2}} \sin(\pi t), \quad \phi(t) = \frac{\sigma_0}{\sqrt{2}} \cos(\pi t). \)

Now let \( \psi(t) = \frac{\sigma_0}{\sqrt{2}} \sin(\pi t), \quad \phi(t) = \frac{\sigma_0}{\sqrt{2}} \cos(\pi t). \)

Note that \( \psi(t) = \psi(t) + \psi(t) + \psi(t) + \psi(t). \)

\textbf{Now, take projection on } \psi(t) \text{ by mult. (1)} \psi(t) \frac{\sigma_0}{\sqrt{2}} \sin(\pi t) \text{ by } \phi(t), \text{ and } \int \psi(t) \phi(t) \, dt.

\textbf{Look at } \psi(t) \text{ coordinates:}

\( \psi(t) = \frac{\sigma_0}{\sqrt{2}} \sin(\pi t), \quad \phi(t) = \frac{\sigma_0}{\sqrt{2}} \cos(\pi t). \)

\textbf{Test IS:} \( \mathbf{A} = \begin{pmatrix} 1 & \psi(t) \phi(t) \end{pmatrix} \mathbf{B}, \quad \mathbf{B} = \frac{\sigma_0}{\sqrt{2}} \sin(\pi t), \quad \mathbf{C} = \frac{\sigma_0}{\sqrt{2}} \cos(\pi t). \)

\textbf{Substitution:} \( \mathbf{L} \mathbf{A} = \begin{pmatrix} 1 & \psi(t) \phi(t) \end{pmatrix} \mathbf{B} = \begin{pmatrix} 1 & \psi(t) \phi(t) \end{pmatrix} \frac{\sigma_0}{\sqrt{2}} \sin(\pi t) \quad \frac{\sigma_0}{\sqrt{2}} \cos(\pi t). \)

\textbf{That IS,} \( \mathbf{L} \mathbf{A} = \begin{pmatrix} 1 & \psi(t) \phi(t) \end{pmatrix} \mathbf{B} = \begin{pmatrix} 1 & \psi(t) \phi(t) \end{pmatrix} \frac{\sigma_0}{\sqrt{2}} \sin(\pi t) \quad \frac{\sigma_0}{\sqrt{2}} \cos(\pi t). \)

\textbf{Match Filter to } \phi(t) \mathbf{A} \text { hote.}
PARRAMOUNT OF TEST 15  

\[ d^2 = \frac{\sigma^2}{d} \left( E_1 + E_2 \cos \theta - 2 \sqrt{E_1 E_2} \cos \phi \right)^2 \]

\[ d^2 = \frac{1}{\sigma^2} \left( E_1 + E_2 \cos \theta - 2 \sqrt{E_1 E_2} \cos \phi \right)^2 \]

\[ \frac{\sigma^2}{d} = \frac{1}{\sigma^2} \left( E_1 + E_2 \cos \theta - 2 \sqrt{E_1 E_2} \cos \phi \right)^2 \]

**Signal Space Interpretation:**

\[ \mathbf{R}(\theta) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \]

**Coordinates:**

- **\( \mathbf{R}_{11} \):** \( \theta_1 \)
- **\( \mathbf{R}_{12} \):** \( \theta_2 \)
- **\( \mathbf{R}_{21} \):** \( \theta_3 \)
- **\( \mathbf{R}_{22} \):** \( \theta_4 \)

**By Law of Cosines,**

\[ d^2 = E_1 + E_2 - 2 \sqrt{E_1 E_2 \cos \theta} \]

Then scale by \( \sigma^2 \).

**If \( \theta = 90^\circ \):**

\[ \mathbf{R}(\phi) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \]

**Coordinates:**

- **\( \mathbf{R}_{11} \):** \( \theta_1 \)
- **\( \mathbf{R}_{12} \):** \( \theta_2 \)
- **\( \mathbf{R}_{21} \):** \( \theta_3 \)
- **\( \mathbf{R}_{22} \):** \( \theta_4 \)

**Summary:**

Regardless of \( \phi \),

\[ \mathbf{R}(\theta) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \]

**Performance:**

\[ \frac{\sigma^2}{d} = \frac{1}{\sigma^2} \left( E_1 + E_2 - 2 \sqrt{E_1 E_2 \cos \theta} \right)^2 \]

Use to find the curve on p. 38.

(In general:

\[ P_0 = \text{Pr} \left( \theta \right), \text{Pr} = \text{Pr} \left( \phi \right) \]

As \( \phi \) varies from 0 to 360.)
Remarkably, the above result holds even if the signals are correlated.

**Special Case 2:** If each hypothesis equally likely a priori \( P \equiv \frac{1}{2} \) and if each signal has the same energy \( \sigma^2 = E \) for \( 1 \leq m \)

Then the optimal receiver is \( \hat{r}_j = \int \hat{r}_j \Rh_i d\Phi(\gamma) \) if largest even if the signals are correlated!

**Proof:** Recall Parseval’s theorem for real orthonormal expansions.

If \( \Rh = \sum_{i=1}^{\infty} \Rh_i \Phi_i(\gamma) \) and \( \Sh = \sum_{i=1}^{\infty} \Sh_i \Phi_i(\gamma) \) then \( \int \Rh_i \Sh_i d\Phi(\gamma) = \sum_{i=1}^{\infty} \Rh_i \Sh_i \)

Then \( \log \frac{1}{2} \sum_{i=1}^{\infty} \Rh_i \Sh_i \) is \( \log \frac{1}{2} \left( \frac{1}{2} \sum_{i=1}^{\infty} \Rh_i \Sh_i + \frac{1}{2} \sum_{i=1}^{\infty} \Rh_i \Sh_i \right) \)

But \( \rho_1 = \sum_{i=1}^{\infty} \Rh_i \Sh_i \) is a \( \gamma \)-invariant measure of \( \Sh \) and \( \Rh = \int \Rh \Ph d\Phi(\gamma) \) where \( \Ph(\gamma) = \frac{1}{\pi} \left( \frac{1}{2} \sum_{i=1}^{\infty} \Rh_i \Sh_i \right) \)

So \( -\frac{1}{\rho_1} \sum_{i=1}^{\infty} \Rh_i \Sh_i \left( \int \Rh \Ph d\Phi(\gamma) \right) = \frac{1}{\pi} \sum_{i=1}^{\infty} \Rh_i \Sh_i \Ph(\gamma) \)

**Note:** This isn’t all that surprising -- we saw it before in the case of general binary detection. Even when \( \Rh = 0 \) and \( \Sh = 0 \) were correlated, the optimal test was \( \int \Rh \Sh d\Phi(\gamma) = \sum_{i=1}^{\infty} \Rh_i \Sh_i \gamma = 0 \) by symmetry if the same energies are equal and a priori probabilities.

See text p. 257

While form of optimal test is unaffected above by correlation between signals, performance of optimal test is affected by correlated signals. For orthogonal signals, best performance.

**Example:** Channel capacity bound (simplied version of text p. 251-256)

**Transmitter:**

- Power \( P \)
- Source of bits \( \text{binomial} B(\alpha) \)
- Encoder: \( \text{group bits into groups of length } \beta \)
- Encode to \( \text{output} \) \( \Ph \)

**Noisy channel:**

- Transmit through noisy channel
- Receiver: \( \text{output} \) \( \hat{r} \)

**Estimate of word:**

- \( \hat{r} \)
- \( \text{estimates of bit} \)

Since all bit choices equally likely, all word choices equally likely.

Each of \( \alpha = 2^{-\beta} \) hypotheses equally likely. Each \( \beta \) has equal energy \( \alpha \).

**Performance:** Let's compute a bound on \( \Pr(\text{error}) \) by symmetry. We say \( \text{let } \beta = \frac{\alpha}{2} \)

\( \Pr(\text{error}) = \sum_{i=2}^{\beta} \beta C_{\beta-i} \alpha^{\beta-i} \left( \frac{1}{2} - \alpha \right) \beta \binom{\beta}{i} \binom{\beta-i}{i} \binom{\beta-i}{i} \)

\( \forall \text{choice of tree, we have } \sum_{i=2}^{\beta} \beta C_{\beta-i} \alpha^{\beta-i} \left( \frac{1}{2} - \alpha \right) \beta \binom{\beta}{i} \binom{\beta-i}{i} \binom{\beta-i}{i} \)

\( \Pr(\text{error}) \leq \sum_{i=2}^{\beta} \beta C_{\beta-i} \alpha^{\beta-i} \left( \frac{1}{2} - \alpha \right) \beta \binom{\beta}{i} \binom{\beta-i}{i} \binom{\beta-i}{i} \)

\( \forall \text{choice of tree, we have } \sum_{i=2}^{\beta} \beta C_{\beta-i} \alpha^{\beta-i} \left( \frac{1}{2} - \alpha \right) \beta \binom{\beta}{i} \binom{\beta-i}{i} \binom{\beta-i}{i} \)

Now let \( \beta = \infty \) (note that \( \alpha = 2^{-\beta} \)).

What happens to this bound?

\( \text{If } \alpha < \frac{1}{2}, \text{ then } \beta < \infty \).

\( \text{For } \alpha > \frac{1}{2}, \text{ we have } \beta = \infty \).

\( \text{Actual channel capacity (approx): } R < H(\alpha) \log_2 \).