INTRODUCTION TO TIME-FREQUENCY DISTRIBUTIONS

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(Preliminary version)

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INTRODUCTION

Q: What are we trying to do?
A: To obtain a "time-local" Fourier transform which can track time-varying spectral properties.

Fourier transform can't track time variations of:

EX #1: $x(t) = \begin{cases} \sin(3t), & \text{if } t < 4; \\ \sin(5t), & \text{if } t > 4. \end{cases}$

EX #2: Chirp $x(t) = e^{j(at)t} = e^{jat^2}$: frequency increases linearly with time.

EX #3: Speech, whale sounds, waves through lossy media.

DESIRED PROPERTIES

$\mathcal{W}_{\tau,\omega}\{x(t)\}$ is spectral density of $x(t)$ for $t \approx \tau$.

We would like the following to hold:

1. $\mathcal{W}_{\tau,\omega}\{x(t)\} \geq 0$ (since it is a density)
2. $\int\int \mathcal{W}_{\tau,\omega}\{x(t)\} d\tau\, d\omega = \int |x(t)|^2 dt$ (total energy conserved)
3. Marginals (more specific energy conservation):
   a. $\int \mathcal{W}_{\tau,\omega}\{x(t)\} d\omega = |x(t)|^2$ (energy at time t)
   b. $\int \mathcal{W}_{\tau,\omega}\{x(t)\} dt = |X(\omega)|^2$ (energy at freq ω)
4. Shifts in time and frequency properly represented:
   a. $\mathcal{W}_{\tau,\omega}\{x(t-t_0)\} = \mathcal{W}_{\tau-t_0,\omega}\{x(t)\}$
   b. $\mathcal{W}_{\tau,\omega}\{x(t)e^{j\omega_0 t}\} = \mathcal{W}_{\tau,\omega-\omega_0}\{x(t)\}$

PROBLEMS

1. Shorter time window→poor freq resolution.
2. Longer window→can't track nonstationarities.
3. Must trade off time and frequency resolution.

BASICS

1. **Densities:** A density is something you *integrate* to get the quantity of interest:
   a. Energy spectral density $|X(\omega)|^2$
   $\rightarrow$energy in band $[\omega_0, \omega_0 + \delta]$ is $\frac{2}{2\pi}|X(\omega_0)|^2\delta$.
   b. We really want "time-freq *density*," not "distribution."

2. **Time-bandwidth product:**
   Regard $|x(t)|^2 \geq 0$ as a prob. density func. for t and also regard $|X(\omega)|^2 \geq 0$ as a pdf for ω.
   a. Δt=duration=σ_t
   b. $\Delta\omega$=bandwidth=σ_ω
   c. Then $(\Delta t)(\Delta\omega) \geq \frac{1}{2}$; equality for Gaussian.

3. $x(t) = A(t)e^{j\phi(t)} \rightarrow A(t)$=amplitude; $\phi(t)$=phase.
   a. $\frac{d\phi}{dt}$ =instantaneous frequency.
   b. Can't define these from $Re[x(t)]$ only. Leads to:

4. **Analytic signal** of $x(t)$ is $x_a(t) = x(t) - j\mathcal{H}\{x(t)\}$.
   a. *Hilbert xform* $\mathcal{H}\{x(t)\} = x(t) * \frac{-1}{\pi t}$
   Transfer function: $jSGN[\omega] \leftrightarrow \frac{\pi}{2}$ phase shift.
   b. $X_a(\omega) = \begin{cases} 2X(\omega), & \text{if } \omega > 0; \\ 0, & \text{if } \omega < 0. \end{cases}$
   c. EX: $x(t) = \cos(at) \rightarrow x_a(t) = e^{jat}$ (quadrature)
   d. $x(t)$ causal$\rightarrow Im[X(\omega)] = \mathcal{H}\{Re[X(\omega)]\}$.

THE CONTINUOUS (MORLET) WAVELET TRANSFORM

Def'n: Continuous or *Morlet* wavelet xform:
$\mathcal{W}_b^a\{x(t)\} = \int x(t)\frac{1}{\sqrt{a}}\psi^*(\frac{t-b}{a})dt, a > 0$

1. Large $a >> 0 \rightarrow$long time scale, coarse resolution.
   Small $0 < a << 1 \rightarrow$short time scale, fine resolution.
2. *Requirements* on the *mother wavelet* $\psi(t)$:
   a. $\Psi(0) = 0 \rightarrow$bandpass (important requirement).
   b. $C = \int_0^\infty \frac{|\Psi(\omega)|^2}{|\omega|}d\omega < \infty$.
   This is satisfied in practice if $\Psi(0) = 0$
   and if $\lim_{|\omega|\rightarrow\infty} \Psi(\omega) = 0 \rightarrow$no impulse in $\psi(t)$.
3. *Reconstruct* $x(t)$ from $\mathcal{W}_b^a\{x(t)\}$ using the formula
   $x(t) = \frac{1}{C}\int_0^\infty \int \mathcal{W}_b^a\{x(t)\}\frac{1}{\sqrt{a}}\psi(\frac{t-b}{a})\frac{db\,da}{a^2}$
   Proof: See V&K p.302-3.
4. **Properties:**
   a. Shift-invariant: $\mathcal{W}_b^a\{x(t-t_0)\} = \mathcal{W}_{b-t_0}^a\{x(t)\}$.
   b. Scale: $\mathcal{W}_b^a\{x(t/c)\} = \sqrt{|c|}\mathcal{W}_{b/c}^{a/c}\{x(t)\}$.
   c. Parseval: $\int |x(t)|^2 dt = \frac{1}{C}\int\int |\mathcal{W}_b^a\{x(t)\}|^2 \frac{da\,db}{a^2}$.
   Proof: See V&K p.306-7.
5. Morlet wavelet: $\psi(t) = \frac{1}{2\pi}e^{-t^2/2}e^{-j\omega_0 t}; \Psi(\omega) = e^{-(\omega-\omega_0)^2/2}$.
   $\omega_0 = 5.336 \rightarrow$1st peak of $Re[\psi(t)]$=half $t = 0$ value.
6. **Sample:** $a = 2^m, b = n2^m \rightarrow$dyadic wavelet sampling.
   This xform is heavily overdetermined and redundant.

Localization Properties of Morlet Wavelet

Let the mother wavelet have duration δt
and bandpass spectrum $[\omega_0 - \frac{\delta\omega}{2}, \omega_0 + \frac{\delta\omega}{2}]$:

1. $a << 1$ $\rightarrow$only "see" $x(t)$ for $b - \frac{a\delta t}{2} < t < b + \frac{a\delta t}{2}$.
   a. Narrow interval$\rightarrow$can track nonstationarity (vary b).
   b. Good time localization; "see" fine details of $x(t)$.
   c. "Zoom-in" on details of $x(t)$.
2. $a << 1$ $\rightarrow$"see" $X(\omega)$ for $\frac{\omega_0}{a} - \frac{\delta\omega}{2a} < \omega < \frac{\omega_0}{a} + \frac{\delta\omega}{2a}$.
   a. High and broad frequency band.
   b. Octave-band filtering (constant $\frac{width}{center}$).

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3. $a >> 1$ $\rightarrow$"see" $x(t)$ for $b - \frac{a\delta t}{2} < t < b + \frac{a\delta t}{2}$.
   a. Broad interval$\rightarrow$can't track nonstationarity.
   b. Little time localization; "see" coarse details of $x(t)$.
   c. "Zoom-out" on $x(t)$; see "big picture."
4. $a >> 1$ $\rightarrow$only "see" $X(\omega)$ for $\frac{\omega_0}{a} - \frac{\delta\omega}{2a} < \omega < \frac{\omega_0}{a} + \frac{\delta\omega}{2a}$.
   a. Low and narrow frequency band.
   b. Octave-band filtering.

THE SHORT-TIME FOURIER TRANSFORM (STFT)

Def'n: The STFT is defined as
$STFT_{\omega,\tau}\{x(t)\} = \int w(t-\tau)x(t)e^{-j\omega t}dt.$

1. Fourier xform of *windowed* (by $w(t-\tau)$) $x(t)$.
   As τ changes, pick off $x(t)$ at different times.
2. *Requirements* on the window $w(t)$: None.
   Usually normalize $\int |w(t)|^2 dt = 1$.
   Should be localized in time and frequency to be useful.
3. *Reconstruct* $x(t)$ from $STFT_{\omega,\tau}\{x(t)\}$:
   $x(t) = \frac{1}{2\pi}\int\int STFT_{\omega,\tau}\{x(t)\}w(t-\tau)e^{j\omega t}d\omega\, d\tau.$
4. **Properties:**
   a. STFT time-frequency tilings all same size.
      Wavelet: t-f tilings have different sizes.
   b. Parseval: $\int |x(t)|^2 dt = \frac{1}{2\pi}\int\int |STFT_{\omega,\tau}\{x(t)\}|^2 d\omega\, d\tau.$
      Proof: V&K p.313-4.
   c. Spectrogram: $|STFT_{\omega,\tau}\{x(t)\}|^2$ is local psd.
5. Gabor function: $w(t) = be^{-at^2}; \quad W(\omega) = b\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}.$
   Best localization in time and frequency.
6. **Sample:** $\omega = m\omega_0, \tau = n\tau_0$
   since time-frequency tilings all have same size.
   STFT is heavily overdetermined and redundant.

TILINGS OF T-F PLANE: STFT AND MORLET

For both sampled STFT and Morlet wavelet xforms:
Express $x(t)$ using basis functions localized in time and freq.

STFT:

1. Basis funcs=$\{w(t-n\tau_0)e^{jm\omega_0 t}, n, m \in integers\}$.
2. $w(t)$=window localized in time and frequency$\rightarrow$
   $w(t-n\tau_0)e^{jm\omega_0 t}$ centered at $(t,\omega)=(n\tau_0, m\omega_0)$.
3. No scaling of t-f plane tiles.
4. $STFT_{\omega,\tau}\{x(t)\}$ projects $x(t)$ onto t-f tiles.

Continuous (Morlet) Wavelet Transform

1. Basis funcs=$\{2^{-m/2}\psi(2^{-m}t-n), n, m \in integers\}$.
2. $\psi(t)$=basis localized in time and frequency$\rightarrow$
   $\psi(2^{-m}t-n)$ centered at $(t,\omega)=(2^m n, 2^{-m}\omega_0)$.
3. Now t-f plane tiles are scaled by $2^{\pm m}$.
4. Wavelet xform projects $x(t)$ onto t-f tiles.

See next page for figures of t-f tilings.

SPECTROGRAM PROPERTIES

DEF: The *spectrogram* is $|STFT_{\omega,\tau}\{x(t)\}|^2$.

1. $Spectrogram = |\int w(t-\tau)x(t)e^{-j\omega t}dt|^2$
2. $= \frac{1}{2\pi}|\int W(\omega-\omega')X(\omega')e^{j\omega'\tau}d\omega'|^2$
   using (generalized) Parseval's theorem
   and also $\mathcal{F}\{w(t-\tau)\} = W(\omega)e^{-j\tau\omega}$.
3. Can view as time-varying local energy spectral density.

Properties

1. Tradeoff between time and freq resolutions:
   a. Narrow window→good time localization,
      poor freq resolution (based on too little data).
   b. Spectral peaks convolved with $W(\omega)$ →blurred.
   c. Also, short time support→large bandwidth.
   d. Wide window→good frequency resolution, poor time localization (signal varies *within* window).
   e. Can see these explicitly in above formulae: tradeoff between compactness of $w(t)$ and $W(\omega)$ (time-bandwidth product).
2. Simple and actually does work fairly well.
3. Can use signal-dependent window $w(t)$.
4. Does NOT satisfy marginals

THE WIGNER-VILLE T-F DISTRIBUTION

Def'n: Wigner-Ville distribution is defined as

$W_{\omega,\tau}\{x(t)\} = \frac{1}{2\pi}\int x(\tau + t/2)x^*(\tau - t/2)e^{-j\omega t}dt.$

1. This is 2-D Fourier *dual* of *ambiguity function.*
2. Dates back to 1932; used in quantum mechanics!
3. $x(\tau + t/2)x^*(\tau - t/2) = instantaneous\, autocorrelation$ of $x(t)$ having lag t at time τ.
   So this is instantaneous version of $S_x(\omega) = \mathcal{F}\{R_x(t)\}$.
4. *Reconstruct* $x(t)$ from $W_{\omega,\tau}\{x(t)\}$:
   $x(t)x^*(0) = \int\int W_{\omega,\tau=t/2}\{x(t)\}e^{j\omega t}d\omega\, d\tau.$
5. **Nice Properties:**
   a. *Marginals are preserved:* If $\mathcal{F}\{x(t)\} = X(\omega)$,
   $\int W_{\omega,\tau}\{x(t)\}d\tau = |X(\omega)|^2$; $\int W_{\omega,\tau}\{x(t)\}d\omega = |x(\tau)|^2$.
   Compare to spectrogram: Total energy conserved:
   $\int\int |STFT_{\omega,\tau}\{x(t)\}|^2 d\omega\, d\tau = \int |x(t)|^2 dt$
   but marginals *not* conserved separately.
   b. *Time and frequency shift-invariant:*
   $W_{\omega,\tau}\{x(t - t_0)e^{j\omega_0 t}\} = W_{\omega-\omega_0,\tau-t_0}\{x(t)\}$
   c. Sharp resolution: For chirp $x(t) = e^{j(at)t} = e^{jat^2}$
   (this is linearly increasing frequency $\omega = at$)
   $W_{\omega,\tau}\{x(t)\} = \delta(\omega - at)$ (try it!)
6. **Bad Properties:**
   a. *Not non-negative:* Can have $W_{\omega,\tau}\{x(t)\} < 0$!
   Very bad for a purported energy spectral density!
   b. *Cross terms:* Since nonlinear, no superposition!
   Solution: Use RID (reduced interference distribution) of Choi-Williams (insert kernel which windows in t-f).

More Properties of Wigner Distribution

1. $x(t) = A(t)e^{j\phi(t)}$

   →average freq at t is $\frac{\int \omega W_{\omega,\tau}\{x(t)\}d\omega}{\int W_{\omega,\tau}\{x(t)\}d\omega} = \frac{d\phi}{dt}$

   =instantaneous frequency, as it should!

2. Product: $W_{\omega,\tau}\{x_1(t)x_2(t)\} = \int W_{\omega',\tau}\{x_1(t)\}W_{\omega-\omega',\tau}\{x_2(t)\}d\omega'$

   Product in time→convolution in freq.

3. Convol: $W_{\omega,\tau}\{x_1(t)*x_2(t)\} = \int W_{\omega,\tau'}\{x_1(t)\}W_{\omega,\tau-\tau'}\{x_2(t)\}d\tau'$

   Product in freq→convolution in time.

4. Example of cross terms: $x(t) = e^{j\omega_1 t} + e^{j\omega_2 t} \rightarrow$

$$W_{\omega,\tau}\{x(t)\} = \delta(\omega-\omega_1)+\delta(\omega-\omega_2)+2\delta(\omega-\tfrac{\omega_1+\omega_2}{2})\cos(\omega_2-\omega_1)t$$

5. Can compute in frequency domain as

   $W_{\omega,\tau}\{x(t)\} = \int x(\tau+t/2)x^*(\tau-t/2)e^{-j\omega t}dt$

   $= \frac{1}{2\pi}\int X(\omega-\theta/2)X^*(\omega+\theta/2)e^{-j\tau\theta}d\theta$

   using (generalized) Parseval's theorem on $\mathcal{F}_{t\rightarrow\theta}$

   and also $\mathcal{F}\{x(t-\tau)\} = X(\theta)e^{-j\tau\theta}$.

CHOI-WILLIAMS (RID) T-F DISTRIBUTION

Def'n: The Choi-Williams t-f distribution,
a type of RID (Reduced Interference Distribution):
$CW_{\omega,\tau}\{x(t)\} = \int\int \frac{1}{|t|}k(\frac{u-\tau}{t})x(u+t/2)x^*(u-t/2)e^{-j\omega t}du\,dt.$
where the *kernel* $k(t) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-t^2/(2\sigma^2)}$

1. **Advantages:**
   a. Kernel $k(t)$ tends to suppress cross terms, at the price of reducing resolution.
   b. Marginals still satisfied for any σ.
2. Can trade these off by varying σ:
   a. $\sigma \to 0$: $k(t) \to \delta(t) \to$Wigner$\to$cross terms.
   b. $\sigma \to \infty$: $k(t) \to k \to$no cross terms, no resolution!
3. Other choices of kernel $k(t)$ can be made.

EXAMPLES: SUPPRESSION OF CROSS TERMS

Example 1: $x(t) = e^{j1t} + e^{j9t}$
(a): Wigner (b): Choi-Williams $\sigma = 0.001$ (c): $\sigma = 0.003$

Example 2: $x(t) = e^{(a_1+jb_1)t^2/2+j\omega_1 t} + e^{(a_2+jb_2)t^2/2+j\omega_2 t}$
(sum of 2 chirps; instantaneous freqs $\omega = b_i t + \omega_i, i = 1, 2$)
(a): Wigner (b): Choi-Williams

COHEN'S CLASS Of TFDS USING KERNELS

IDEA (L. Cohen): Define time-freq dist. (TFD) as:

1. Define *ambiguity function* $A_{\theta,\tau}\{x(t)\}$ as

   $A_{\theta,\tau}\{x(t)\} = \int x(u+\tau/2)x^*(u-\tau/2)e^{j\theta u}du.$

   Note $4\pi^2 W_{\omega,t}\{x(t)\} = \mathcal{F}\mathcal{F}\{A_{\theta,\tau}\{x(t)\}\}$
   $= \int\int A_{\theta,\tau}\{x(t)\}e^{-j\theta t}e^{-j\omega\tau}d\theta\, d\tau.$

2. Define $W'_{\omega,t}\{x(t)\}$ as

   $W'_{\omega,t}\{x(t)\} = \frac{1}{4\pi^2}\int\int k(\theta,\tau)A_{\theta,\tau}\{x(t)\}e^{-j\theta t}e^{-j\omega\tau}d\theta\, d\tau$
   for some *kernel* $k(\theta,\tau)$.

   Note this convolves $W_{\omega,\tau}\{x(t)\}$ with $\mathcal{F}^{-1}\mathcal{F}^{-1}\{k(\theta,\tau)\}$.

Examples:

1. $k(\theta,\tau) = 1 \rightarrow$Wigner-Ville (easy).
2. $k(\theta,\tau) = e^{-\theta^2\tau^2/\sigma} \rightarrow$Choi-Williams using

   $\int e^{j\theta(u-t)}e^{-\theta^2\tau^2/\sigma}d\theta = \frac{\sqrt{\sigma\pi}}{\tau}e^{-\frac{\sigma}{4\tau^2}(u-t)^2}$

3. Other choices possible (see below).
4. To satisfy marginals, need $k(0,\tau) = k(\theta,0) = 1$.
   To satisfy energy conservation, need $k(0,0) = 1$.
5. Kernel can be made signal-dependent (Baraniuk-Jones).
6. Recover signal from $W'_{\omega,t}\{x(t)\}$ using

   $2\pi s(t)s^*(0) = \int\int\int \frac{W'_{\omega,u}\{x(t)\}}{k(\theta,\tau)}e^{j\omega t}e^{j\theta u}e^{-j\theta t/2}du\, d\omega\, d\theta$

   For spectrogram, how can we recover signal from $|\mathcal{F}|$?
   Is magnitude of STFT, not $\mathcal{F}$!

RECONSTRUCTION OF DISPERSIVE LAYERED MEDIA

(Chien and Yagle, 1989)

Given:

1. Dispersive (frequency-dependent absorption) layered (absorption, density vary with depth) medium.
2. Reflection response $R(t)$ of medium to an impulse (deconvolved from actual source explosion signal)

Goal:

To reconstruct density ρ_i and absorption Q_i in i^{th} layer of acoustic medium for all i.

Assumptions:

1. Absorption factor$=e^{-\frac{|\omega|}{2cQ_i}z}$
   z=distance in i^{th} layer, c=wave speed.
2. Reflection coefficient $r_i = \frac{\rho_i c_i - \rho_{i+1} c_{i+1}}{\rho_i c_i + \rho_{i+1} c_{i+1}}$.
3. Neglect multiple reflections.
4. Layers equally thick (use very thin layers).

Approach:

1. Compute time-freq distribution of $R(t)$.
2. Window tfd in time to separate layer reflections.
3. Intervals between layer reflections$\rightarrow c_i$.
4. Fit exponential curve to freq variation at each time.
   "Time constant"$=2\sum_i Q_i c_i$ (actually fit line to log).
5. Amplitude of tfd (intercept)$\rightarrow r_i \rightarrow \rho_i$.

TRACKING NON-CONSTANT BLOOD FLOW IN MRI

(Sahiner and Yagle, 1989)

Given:

1. Moving object (e.g., blood) undergoing MRI.
   Motion blurs MRI signal; must deconvolve it.
2. MRI signal of moving object.
   Both motion and object are unknown.

Goal:

To reconstruct both the unknown velocity and the unknown (nuclear) spin density

Assumptions:

1. Everything occurs in one direction (projection).
2. Velocity affine function of location
   a. Reasonable for thin MRI slice excitation of steady-state flow in blood vessel.
   b. Use different functions in each slice→ trapezoidal approx. to velocity vs. location.
3. $\hat{\rho}(\frac{k}{v_1}(e^{v_1 t}-1)) \approx$constant.

Approach:

1. Compute time-freq distribution of MRI signal

   $$x(t) = e^{j2\pi k \frac{v_0}{v_1^2}(e^{v_1 t} - v_1 t - 1)} \hat{\rho}(\frac{k}{v_1}(e^{v_1 t}-1))$$

   velocity $v(x) = v_0 + v_1 x$, $\hat{\rho}(k) = \mathcal{F}\{spin\,density(x)\}$.
2. Instant freq$=\frac{d\phi}{dt} = \frac{kv_0}{v_1}(e^{v_1 t}-1)$
3. Trajectory of tfd maxima→ v_0, v_1.