SPLINES AND BATTLE-LEMARIE WAVELET

Splines: Piecewise polynomial approximation to functions:
1. Divide up real line into intervals;
2. Within interval, use polynomial of degree \( k \);
3. At boundaries (called knots) have continuous \( \frac{d^{k-1}}{dt^{k-1}} \).

Basis for spline spaces: B-splines (B-s)

V&K notation is awful. Summary of notation:

<table>
<thead>
<tr>
<th>source</th>
<th>( B-s )</th>
<th>( \mathcal{F}{ B-s } )</th>
<th>( \mathcal{F}{ \text{sampled } B-s } )</th>
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</thead>
<tbody>
<tr>
<td>V&amp;K</td>
<td>( \beta^{(N)}(t) )</td>
<td>( B^{(N)}(\omega) )</td>
<td>( B^{(N)}(\omega) )</td>
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<tr>
<td>aey</td>
<td>( s_N(t) )</td>
<td>( S_N(\omega) )</td>
<td>( \tilde{S}_N(\omega) )</td>
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Definitions

**B-spline order** 2\( N-1 \): \( S_{2N-1}(\omega) = \left( \frac{\sin(\omega/2)}{\omega/2} \right)^{2N} \)

**B-spline order** 2\( N \): \( S_{2N}(\omega) = e^{-j\omega/2} \left( \frac{\sin(\omega/2)}{\omega/2} \right)^{2N+1} \)

Note even-order splines shifted by 1/2 in time so that \( s_n(t) \neq 0 \) for \( 0 < t < N + 1 \).

\( s_0(t) \)=pulse, \( s_1(t) \)=triangle, \( s_2(t) \)=quadratic, etc.

\( V_k^N = \{ \text{piecewise polynomials of degree } N \text{ over intervals } [2^k j, 2^k (j+1)), j \in \text{integers} \text{ having continuous } \frac{d^{k-1}}{dt^{k-1}} \text{ at } t = 2^k j \} \).

Have chain of subspaces \( \ldots V_1^N \subset V_0^N \subset V_{-1}^N \subset \ldots \)
\( \rightarrow s_N(t) \) satisfies 2-scale equation.

\( x(t) \) orthogonal to its integer translations
\( \leftrightarrow \int x(t)x(t-j)dt = \text{sampled autocorrelation} = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if } j \neq 0 \end{cases} \)
\( \leftrightarrow \sum_{k=-\infty}^{\infty} |X(\omega + 2k\pi)|^2 = 1 \)
(sampling induces periodicity in \( \omega \))
\{s_N(t - j), j \in \text{integers}\} \text{ spans spline space } V_0^N. \\
But \{s_N(t - j)\} \text{ not orthonormal basis for } V_0^N.

Let \(\tilde{S}_{2N+1}(\omega) = \sum_{k=-\infty}^{\infty} |S_N(\omega + 2k\pi)|^2\). Then:

1. \(\tilde{S}_{2N+1}(\omega)\) periodic\(\rightarrow\)discrete in time
2. \(\tilde{S}_{2N+1}(\omega) = DTFT[sampled \ s_{2N+1}(t)]\)
   since \(s_N(t) * s_N(t) = s_{2N+1}(t)\) and \(s_N(t)\) time-symmetric.

Let \(\Phi(\omega) = S_N(\omega) / \sqrt{\tilde{S}_{2N+1}(\omega)}\)=scaling function.
Then \(\sum_{k=-\infty}^{\infty} |\Phi(\omega + 2k\pi)|^2 = 1\)
\(\rightarrow \phi(t)\) orthogonal to its translations
\(\rightarrow \{\phi(t - j)\} \text{ orthonormal basis for } V_0^N.\)

\(s_N(t)\) satisfies 2-scale equation\(\rightarrow \phi(t)\) does also.
Above\(\rightarrow \phi(t)\) is a wavelet scaling function.
\(\phi(t)\) longer has compact support; decays exponentially.

\begin{center}
**Battle-Lemarie Wavelets:**
\end{center}
1. Algebraic details for \(s_1(t)\) on p.233-234 of V&K.
2. \(\phi(t)\) longer compact support; decays exponentially.
3. Can also use \(\phi(t) = s_N(t)\): lose intrascale orthogonality
   while keeping interscale orthogonality.
   Advantage: \(s_N(t)\) localized in time and frequency.