THE CONTINUOUS (MORLET) WAVELET TRANSFORM

Def'n: Continuous or *Morlet* wavelet xform:
$\mathcal{W}_b^a\{x(t)\} = \int x(t) \frac{1}{\sqrt{a}} \psi^*(\frac{t-b}{a}) dt, a > 0$

1. Large $a >> 0$ →long time scale, coarse resolution.
   Small $0 < a << 1$ →short time scale, fine resolution.
2. *Requirements* on the *mother wavelet* $\psi(t)$:
   a. $\Psi(0) = 0$ →bandpass (important requirement).
   b. $C = \int_0^\infty \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$.
   This is satisfied in practice if $\Psi(0) = 0$
   and if $\lim_{|\omega| \to \infty} \Psi(\omega) = 0$ →no impulse in $\psi(t)$.
3. *Reconstruct* $x(t)$ from $\mathcal{W}_b^a\{x(t)\}$ using the formula
   $x(t) = \frac{1}{C} \int_0^\infty \int \mathcal{W}_b^a\{x(t)\} \frac{1}{\sqrt{a}} \psi(\frac{t-b}{a}) \frac{db\, da}{a^2}$
   Proof: See V&K p.302-3.
4. **Properties:**
   a. Shift-invariant: $\mathcal{W}_b^a\{x(t - t_0)\} = \mathcal{W}_{b-t_0}^a\{x(t)\}$.
   b. Scale: $\mathcal{W}_b^a\{x(t/c)\} = \sqrt{|c|} \mathcal{W}_{b/c}^{a/c}\{x(t)\}$.
   c. Parseval: $\int |x(t)|^2 dt = \frac{1}{C} \int \int |\mathcal{W}_b^a\{x(t)\}|^2 \frac{da\, db}{a^2}$.
   Proof: See V&K p.306-7.
5. Morlet wavelet: $\psi(t) = \frac{1}{2\pi} e^{-t^2/2} e^{-j\omega_0 t}; \Psi(\omega) = e^{-(\omega-\omega_0)^2/2}$.
   $\omega_0 = 5.336$ →1st peak of $Re[\psi(t)]$=half $t = 0$ value.
6. **Sample:** $a = 2^m, b = n2^m$ →dyadic wavelet sampling.
   This xform is heavily overdetermined and redundant.

THE SHORT-TIME FOURIER TRANSFORM (STFT)

Def'n: The STFT is defined as
$STFT_{\omega,\tau}\{x(t)\} = \int w(t-\tau)x(t)e^{-j\omega t}dt.$

1. Fourier xform of *windowed* (by $w(t-\tau)$) $x(t)$.
   As τ changes, pick off $x(t)$ at different times.
2. *Requirements* on the window $w(t)$: None.
   Usually normalize $\int |w(t)|^2 dt = 1$.
   Should be localized in time and frequency to be useful.
3. *Reconstruct* $x(t)$ from $STFT_{\omega,\tau}\{x(t)\}$:
   $x(t) = \frac{1}{2\pi}\int\int STFT_{\omega,\tau}\{x(t)\}w(t-\tau)e^{j\omega t}d\omega\, d\tau.$
4. **Properties:**
   a. STFT time-frequency tilings all same size.
      Wavelet: t-f tilings have different sizes.
   b. Parseval: $\int |x(t)|^2 dt = \frac{1}{2\pi}\int\int |STFT_{\omega,\tau}\{x(t)\}|^2 .d\omega\, d\tau$
      Proof: V&K p.313-4.
   c. Spectrogram: $|STFT_{\omega,\tau}\{x(t)\}|^2$ is local psd.
5. Gabor function: $w(t) = be^{-at^2}; \quad W(\omega) = b\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}.$
   Gabor logon: basis function $be^{-a(t-\tau)^2}e^{j\omega t}$.
   Best localization in time and frequency.
6. **Sample:** $\omega = m\omega_0, \tau = n\tau_0$
   since time-frequency tilings all have same size.
   STFT is heavily overdetermined and redundant.