STOCHASTIC FRACTALS USING WAVELETS

DEF: A *stochastic fractal* is a *self-similar* random process $x(t)$
Wide-sense self-similar: Self-similar mean and covariance
$E[x(t)] = a^{-H}E[x(at)]; \quad E[x(t_1)x(t_2)] = a^{-2H}E[x(at_1)x(at_2)].$

Strict-sense self-similar: $x(t), a^{-H}x(at)$ same joint pdfs.
Gaussian process: strict-sense=wide-sense. Assume in sequel.

Examples of Stochastic fractal processes:

1. White Gaussian processes: $H = -1/2$ since $a\delta(at) = \delta(t)$
2. Wiener process=integrated white Gaussian process =continuous-time random walk=Brownian motion:

   $$E[x(t_1)x(t_2)] = \sigma^2 MIN[t_1, t_2] = a^{-1}\sigma^2 MIN[at_1, at_2].$$

3. $1/f$ *processes*: $S_x(\omega) = \sigma^2/|\omega|^{2H+1}$
   where $S_x(\omega) = \mathcal{F}\{R_x(\tau)\}$ and $R_x(\tau) = E[x(t)x(t-\tau)]$.

   a. Valid over decades of ω. Sample paths fractals.
   b. Stock market indices; EEGs; EKGs; river levels; noise
   c. **Wiener process:** $2H + 1 = 2\frac{1}{2} + 1 = 2$ makes sense: integrate white process $\rightarrow S_x(\omega) = 1/|\omega|^2$.

Properties of 1/f processes:

1. **Time domain:** Let $w(t)$ be 0-mean white Gaussian. Then:

$$x(t) = \frac{1}{\Gamma(H+\frac{1}{2})}[\int_{-\infty}^{t} |t-\tau|^{H-\frac{1}{2}} w(\tau)d\tau - \int_{-\infty}^{0} |\tau|^{H-\frac{1}{2}} w(\tau)d\tau].$$

   a. Second term$\rightarrow$stable system (Barton and Poor).
   b. Fractal dimension of sample paths$=2-H, 0 < H < 1$
   c. This $x(t)$ has stationary self-similar increments.
   d. $H = 1/2 \rightarrow$usual Wiener process. In fact:

$$E[x(t_1)x(t_2)] = [\Gamma(1-2H)\frac{\cos(\pi H)}{2\pi H}][|s|^{2H} + |t|^{2H} - |t-s|^{2H}].$$

2. Some Fourier and Laplace relations:
   a. $\mathcal{L}\{\frac{1}{\Gamma(H+\frac{1}{2})}t^{H-\frac{1}{2}}1(t)\} = \frac{1}{s^{H+\frac{1}{2}}}$, $1(t)$=unit step.
   b. $\mathcal{F}^{-1}\{\frac{1}{|\omega|^{2H+1}}\} = |t|^{2H}\frac{1}{2\Gamma(2H+1)\cos((H+\frac{1}{2})\pi)}$
3. Not integrable→not really a valid power spectral density.
   a. $H < 0$ →slow rolloff as $\omega \to \infty$ →"UV catastrophe." Non-integrable over ω →high-frequency power.
   b. $H > 0$ →blows up as $\omega \to 0$ →"infrared catastrophe." Non-integrable near origin→suggests non-stationarity.
4. **DEF** (Wornell): $x(t)$ is $1/f$ process if *bandpass filtering* $\to S_x(\omega) = \sigma^2/|\omega|^{2H+1}$ in the pass band.

Characterization of fractal processes using wavelets:

1. Let x_n^m be 0-mean uncorrelated random variables with $\sigma^2_{x_n^m} = 2^{-(2H+1)m}\sigma^2$.

Then $x(t) = \sum\sum x_n^m 2^{m/2}\psi(2^m t - n)$ has time-averaged psd

$S_x(\omega) = \sigma^2 \sum 2^{-(2H+1)m}|\Psi(2^{-m}\omega)|^2 \to S_x(\omega) = 2^{k(2H+1)}S_x(2^k\omega)$

which is self-similar between octaves.

   a. Time-averaging needed since $x(t)$ nonstationary.
   b. $\psi(t)$ must be regular: $\Psi(\omega) \simeq 1/|\omega|^{H+\frac{1}{2}}$.
   c. $H = -\frac{1}{2}$ →white process ($\sigma^2_{x_n^m}$=constant; $S_x(\omega) = 1$)

2. Given $1/f$ process $x(t)$ and $x_n^m = \int x(t)2^{m/2}\psi(2^m t - n)dt$,

$E[x_n^m x_{n'}^{m'}] = \int \frac{2^{-(m+m')/2}\sigma^2}{2\pi|\omega|^{2H+1}}\Psi(2^{-m}\omega)\Psi^*(2^{-m'}\omega)e^{-j\omega(n2^{-m}-n'2^{-m'})}d\omega$

   a. Stationary at fixed scale $m = m'$ ($n - n'$ only).
   b. $\rho_{n,n'}^{m,m'}$ stationary across scales if $2^{-m}n = 2^{-m'}n'$.
   c. If $\psi(t)$ has M vanishing moments (e.g., Battle-Lemarie), $\rho_{n,n'}^{m,m'} \simeq 1/|2^{-m}n - 2^{-m'}n'|^{2M-2H-1}$.