1. Let $x_i = \begin{cases} 1 & \text{if } i^{th} \text{ toaster is dented (happens with prob. } p) \\ 0 & \text{if } i^{th} \text{ toaster not dented (happens with prob. } 1-p) \end{cases}$. Let $w = \sum_{i=1}^{n} x_i$.

$E[x_i] = 1(p) + 0(1-p) = p$. $E[x_i^2] = 1^2(p) + 0^2(1-p) = p$. Both make sense.

$\sigma_{x_i}^2 = E[x_i^2] - (E[x_i])^2 = p - p^2 = p(1-p)$. Note symmetry between p and $1-p$.

$E[w] = nE[x_i] = np = (2000)(.05) = 100$. $\sigma_w^2 = np(1-p) = (2000)(.05)(.95) = 95$.

$Pr[110 \leq w \leq 2000] = \Phi[\frac{2000+\frac{1}{2}-100}{\sqrt{95}}] - \Phi[\frac{110-\frac{1}{2}-100}{\sqrt{95}}] = 1.000 - 0.835 = 0.165$.

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2. Recognizing (?) $f_{x,y}(X,Y) \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$, we can plug into lecture result.

OR: Know $f_x(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-X^2/(2\sigma^2)} \sim \mathcal{N}(0, \sigma^2)$ and $f_y(Y) \sim \mathcal{N}(0, \sigma^2) \rightarrow E[y] = 0$.

THEN: $f_{y|x}(Y|X) = \frac{f_{x,y}(X,Y)}{f_x(X)} = \frac{1}{\sqrt{2\pi\sigma^2}\sqrt{1-\rho^2}} e^{-(Y-\rho X)^2/[2\sigma^2(1-\rho^2)]}$ after a bit of algebra.

SO: We recognize $f_{y|x}(Y|X) \sim \mathcal{N}(\rho X, \sigma^2(1-\rho^2)) \rightarrow E[y|x=X] = \rho X \rightarrow$least-squares predictor of y from observation $x = X$ is ρX. Note what happens for $\rho = 0$ or ± 1.

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3. $\Phi_{x_i}(\omega) = \int_{-\infty}^{\infty} (\frac{1}{2}\delta(X-1) + \frac{1}{2}\delta(X+1)) e^{j\omega X} dX = \frac{1}{2}(e^{j\omega} + e^{-j\omega}) = \cos\omega$.

$y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \rightarrow \Phi_y(\omega) = \Phi_{x_i}(\frac{\omega}{\sqrt{n}})^n = cos^n(\frac{\omega}{\sqrt{n}}) = e^{n \log\cos(\omega/\sqrt{n})}$ QED.

$\underset{n\to\infty}{\text{LIM}} \Phi_y(\omega) = \underset{n\to\infty}{\text{LIM}} \cos^n(\frac{\omega}{\sqrt{n}}) = \underset{n\to\infty}{\text{LIM}} (1 - \frac{\omega^2}{2n})^n = e^{-\omega^2/2}$ (easier to see here).

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4. Let x_i=#letters received on i^{th} day, $y_n = \sum_{i=1}^{n} x_i$, and $z_n = y_n/2$=integer.

$E[x_i] = \frac{1}{4}(0) + \frac{2}{4}(2) + \frac{1}{4}(4) = 2$. $E[x_i^2] = \frac{1}{4}(0)^2 + \frac{2}{4}(2)^2 + \frac{1}{4}(4)^2 = 6$.

$E[z_n] = \frac{n}{2}E[x_i] = n$. $\sigma_{z_n}^2 = \frac{n}{4}\sigma_{x_i}^2 = \frac{n}{4}(6 - 2^2) = \frac{n}{2} \rightarrow \sigma_{z_n} = \sqrt{n/2}$.

4a. $0.919 = \Phi(1.4) = Pr[y_n \geq 40] = Pr[20 \leq z_n < \infty] = 1 - \Phi[\frac{20-n}{\sqrt{n/2}}] = \Phi[\frac{n-20}{\sqrt{n/2}}]$

$\rightarrow 1.4 = (n-20)/\sqrt{n/2} \rightarrow (n-20) = \sqrt{n} \rightarrow n = 25$ (neglecting Demoivre-Laplace).

4b. $Pr[96 \leq y_{50} \leq 102] = Pr[48 \leq z_{50} \leq 51] = \Phi[\frac{51+\frac{1}{2}-50}{\sqrt{50/2}}] - \Phi[\frac{48-\frac{1}{2}-50}{\sqrt{50/2}}]$

$= \Phi(0.3) - \Phi(-0.5) = \Phi(0.3) + \Phi(0.5) - 1 = 0.618 + 0.691 - 1 = 0.309$.

4c. $Pr[y_n > 2n + \sqrt{2n}] = Pr[z_n > n + \sqrt{n/2}] = Pr[\frac{z_n - n}{\sqrt{n/2}} > 1] = Pr[\frac{z_n - E[z_n]}{\sigma_{z_n}} > 1] \simeq$ $1 - \Phi(1) = 0.159$ (the Demoivre-Laplace correction is negligible for large n).

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5. Let $\hat{S} = \frac{1}{n}\sum_{i=1}^{n} x_i \rightarrow E[\hat{S}] = \frac{nS}{n} = S$ and $\sigma_{\hat{S}}^2 = \frac{nS(1-S)}{n^2} = \frac{S(1-S)}{n} \leq \frac{1}{4n} \left(\begin{smallmatrix}\text{worst} \\ \text{case}\end{smallmatrix} : S = \frac{1}{2}\right)$.

$Pr[|\hat{S} - S| < 0.02] = \Phi[\frac{0.02}{1/\sqrt{4n}}] - \Phi[\frac{-0.02}{1/\sqrt{4n}}] = 2\Phi[\frac{0.02}{1/\sqrt{4n}}] - 1 = 0.95$

$\rightarrow \Phi[0.04\sqrt{n}] = \frac{0.95+1}{2} \rightarrow 0.04\sqrt{n} = \Phi^{-1}[0.975] = 1.96 \rightarrow n = 2401$ (minimum n).

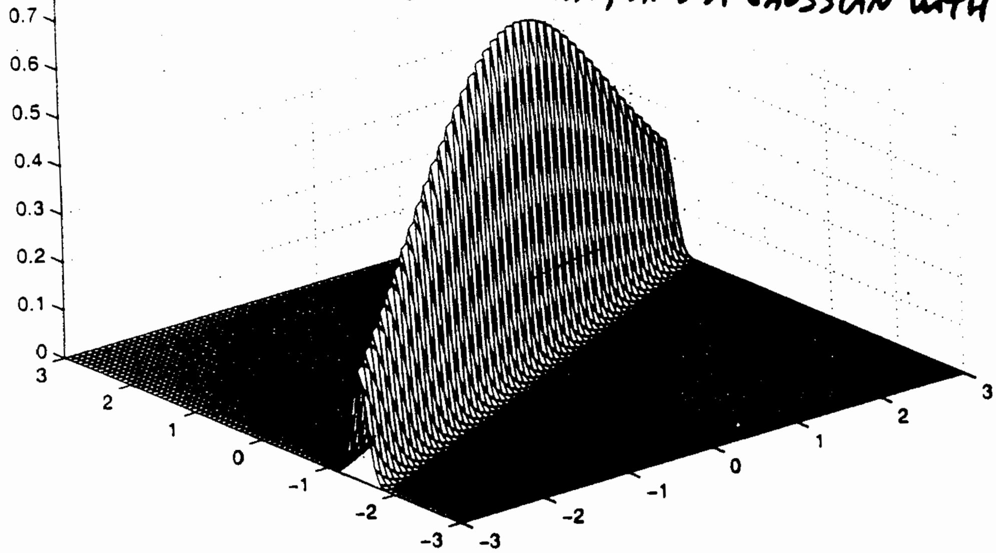
6 (a) COVARIANCE MATRIX OF $\begin{bmatrix} X \\ Y \end{bmatrix}$ IS $K = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ WHICH HAS EIGENVALUES 5, 0 EIGENVECTORS $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (FROM MATLAB OR BY HAND).

THIS IMPLIES $Z = \underline{\phi}_2^T \begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{\sqrt{5}}(X - 2Y)$ HAS VARIANCE $\sigma_Z^2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}\frac{1}{\sqrt{5}} = 0$

$\rightarrow Z = E[Z] = \frac{1}{\sqrt{5}}(E(X) - 2E(Y)) = 0$ WITH PROB. 1. $X = 3 \rightarrow$ $\boxed{Y = 1.5 \text{ WITH PROB. } 1}$

(b) $\begin{bmatrix} Z \\ W \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \underline{\phi}_2^T \\ \underline{\phi}_1^T \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = [\underline{\phi}_2 \mid \underline{\phi}_1]^T \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$

IS A ROTATION BY $-63.4°$. THE JOINT pdf IS AN IMPULSIVE RIDGE IN THE $[1 \ -2]$ DIRECTION, AND A GAUSSIAN WITH VARIANCE $\lambda_1 = 5$ IN THE $[2 \ 1]$ OR $[-2 \ -1]$ DIRECTION:



SINCE K IS SINGULAR, WE CAN'T USE THE EXPRESSION FOR THE 2-D GAUSSIAN pdf (NOTE $\rho = 1$; CAN'T DIVIDE BY $1-\rho^2$!)

(c) NOW $K = \begin{bmatrix} 4.01 & 2 \\ 2 & 1.01 \end{bmatrix}$ WHICH HAS EIGENVALUES 5.01, 0.01 (INCREASE BY 0.01) EIGENVECTORS $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (SAME).

JUST PLUG IN:

$$f_{X,Y}(X,Y) = \frac{1}{2\pi\sqrt{\det K}} e^{-\frac{1}{2}[X \ Y] K^{-1}\begin{bmatrix} X \\ Y \end{bmatrix}} = \boxed{\frac{1}{2\pi(0.0501)^{\frac{1}{2}}} e^{-\left[\frac{1}{2(0.0501)}\left(1.01X^2 + 4.01Y^2 - 4XY\right)\right]}}$$

NOW YOU KNOW WHY I CHOSE THESE NUMBERS!

$\det K = (4.01)(1.01) - (2)(2) = 0.0501$

$$K^{-1} = \frac{\begin{bmatrix} 1.01 & -2 \\ -2 & 4.01 \end{bmatrix}}{0.0501} \approx \begin{bmatrix} 20 & -40 \\ -40 & 80 \end{bmatrix}$$

$$K^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 100 & 0 \\ 0 & \frac{1}{5.01} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

DOESN'T LOOK VERY EXCITING. BUT REWRITE THE EXPONENT AS:

$$-\frac{1}{2}[X \ Y] K^{-1}\begin{bmatrix} X \\ Y \end{bmatrix} = -\frac{1}{2}[X \ Y]\overbrace{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 100 & 0 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\frac{1}{\sqrt{5}}}^{K^{-1}}\Big[$$

$$= -\frac{1}{2}[Z \ W]\begin{bmatrix} 100 & 0 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} Z \\ W \end{bmatrix} = -(50Z^2 + 0.1W^2) \ !$$

QUITE A DIFFERENCE!