1. We are given that $f_{y|\theta}(Y|\Theta) \sim \mathcal{N}(\Theta, \sigma^2) \to E[y|\theta = \Theta] = \Theta$.

   Then $E[y] = E_\theta[E_y[y|\theta = \Theta]] = \int E[y|\theta = \Theta] f_\theta(\Theta) d\Theta = \int_0^{2\pi} \Theta \frac{1}{2\pi} d\Theta = \pi$.

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2. From pp. 138-139, we know that $f_z(Z) = \frac{Z}{\sigma^2} e^{-Z^2/(2\sigma^2)}, Z \geq 0$ (Rayleigh pdf).

   $E[z] = \int_0^\infty Z^2 e^{-Z^2/2} dZ = -Ze^{-Z^2/2}|_0^\infty + \int_0^\infty e^{-Z^2/2} dZ = 0 + \sqrt{\pi/2} = \sqrt{\pi/2}$

   using integration by parts: $\int u\, dv = uv - \int v\, du$ where $u = Z$ and $v = -e^{-Z^2/2}$

   and $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-X^2/2} dX = 1 \to \int_0^\infty e^{-X^2/2} dX = \frac{1}{2}\sqrt{2\pi} = \sqrt{\pi/2}$ by symmetry.

   $E[z^2] = \int_0^\infty Z^3 e^{-Z^2/2} dZ = \int_0^\infty 2Ue^{-U} dU = 2$ where $U = Z^2/2$.

   So $\sigma_z^2 = E[z^2] - (E[z])^2 = 2 - \pi/2 = 0.4292$.

3a. $E_y[E_x[g(x,y)|y]] = \int dY\, f_y(Y) (\int g(X,Y) f_{x|y}(X|Y) dX)$

   $= \int dY \int dX\, g(X,Y) f_y(Y) f_{x|y}(X|Y) = \int dY \int dX\, g(x,y) f_{x,y}(X,Y) = E[g(x,y)]$.

3b. $g(x,y) = x$ and $g(x,y) = x^2 \to \sigma_x^2 = E[x^2] - (E[x])^2 = E_y[E[x^2|y]] - (E_y[E[x|y]])^2$.

   $E_y[\sigma_{x|y}^2] = E_y[E[x^2|y]] - E_y[(E[x|y])^2]$ and $Var_y[E[x|y]] = E_y[(E[x|y])^2] - (E_y[E[x|y]])^2$.

   Adding, we get $\sigma_x^2 = E_y[\sigma_{x|y}^2] + Var_y[E[x|y]]$ Q.E.D. What does this mean? See #4.

4a. $E[y] = E_n[E[y|n]] = E_n[nE[x]] = E[n]E[x]$ since $y = x_1 + x_2 + \ldots + x_n$.

   $\sigma_y^2 = E_n[\sigma_{y|n}^2] + Var_n[E[y|n]] = E_n[n\sigma_x^2] + Var_n[nE[x]] = E[n]\sigma_x^2 + (E[x])^2\sigma_n^2$.

   n constant$\to \sigma_y^2 = n\sigma_x^2$. But x constant$\to \sigma_y^2 = x^2\sigma_n^2$. Makes sense since $y = xn$.

4b. Total number r of items sold=sum of random number (k) of random variables (n).

   $\sigma_r^2 = E[k]\sigma_n^2 + (E[n])^2\sigma_k^2 = \mu_1\mu_2 + \mu_1\mu_2^2$ where $E[k] = \sigma_k^2 = \mu_1$ and $E[n] = \sigma_n^2 = \mu_2$.

   Increasing μ_1, as opposed to μ_2, leads to a smaller increase in σ_r^2.

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5. See overleaf.

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6. Let $A_n = \{\omega : x(\omega) > \frac{1}{n}\}$. $A_1 \subset A_2 \subset \cdots \subset \{\omega : x(\omega) > 0\}$. Continuity of probability

   $\to Pr[x > 0] = Pr[\{\omega : x(\omega) > 0\}] = Pr[\underset{n\to\infty}{\text{LIM}} A_n] = \underset{n\to\infty}{\text{LIM}} Pr[A_n] = \underset{n\to\infty}{\text{LIM}} Pr[x > \frac{1}{n}]$

   $\leq \underset{n\to\infty}{\text{LIM}} nE[x] = 0 \to Pr[x = 0] = 1$ using the Markov inequality and $E[x] = 0$.

5 ORDER STATISTICS

(a) TRANSFORMATION $\underline{Y} = g(\underline{X})$:

Y_1 = SMALLEST $[X_1 \ldots X_n]$
Y_2 = 2ND SMALLEST $(X_1 \ldots X_n)$
$\vdots$
Y_n = LARGEST $[X_1 \ldots X_n]$

INVERSE TRANSFORMATION $\underline{X} = g^{-1}(\underline{Y})$

$X_1 < X_2 < \cdots < X_n \leftrightarrow \begin{bmatrix} Y_1 = X_1 \\ \vdots \\ Y_n = X_n \end{bmatrix}$

$X_2 < X_1 < X_3 < \cdots < X_n \leftrightarrow \begin{bmatrix} Y_1 = X_2 \\ Y_2 = X_1 \\ Y_3 = X_3 \end{bmatrix}$

TOTAL OF $n!$ ORDERINGS
$\rightarrow$ $n!$ DIFFERENT INVERSE MAPPINGS.

JACOBIAN FOR EACH $\underline{x} = g^{-1}(\underline{y})$ IS OBVIOUSLY 1 (JUST REORDERING).

SO $\underline{f_{Y_1 \ldots Y_n}(Y_1 \ldots Y_n) = n!\, f_X(Y_1) \cdots f_X(Y_n) \text{ FOR } Y_1 < \cdots < Y_n}$; 0 OTHERWISE.

DOES THIS INTEGRATE TO 1? WE KNOW $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_X(Y_1) \cdots f_X(Y_n)\, dY_1 \ldots dY_n = 1$.

$\int_{-\infty}^{\infty} \int_{-\infty}^{Y_n} \cdots \int_{-\infty}^{Y_2} f_X(Y_1) \cdots f_X(Y_n)\, dY_1 \ldots dY_n < 1$, BUT $n!$ ORDERINGS OF $\underline{\quad}$, AND EACH IS EQUAL BY SYMMETRY, SO THIS INTEGRATES TO $\frac{1}{n!}$. MULT. BY $n!$ TO MAKE INTEGRATE TO 1.

(b) $f_{Y_k}(Y_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{Y_1 \ldots Y_n}(Y_1 \ldots Y_n)\, \underbrace{dY_1 \ldots dY_n}_{\text{SKIPPING } dY_k}$

NOW, INNERMOST INTEGRAL $\int_{-\infty}^{Y_2} f_X(Y_1) = \underline{F_X(Y_2)}$ BY DEFINITION OF F_X (PDF). NOTE $Y_1 < Y_2$.

2ND INNERMOST INTEGRAL $\int_{-\infty}^{Y_3} F_X(Y_2) f_X(Y_2)\, dY_2 = \int_{-\infty}^{Y_3} F_X(Y_2)\, dF_X(Y_2) = \frac{1}{2}(F_X(Y_3))^2$. NOTE $Y_2 < Y_3$.

3RD INNERMOST INTEGRAL $\int_{-\infty}^{Y_4} \frac{1}{2}(F_X(Y_3))^2 f_X(Y_3)\, dY_3 = \int_{-\infty}^{Y_4} \frac{1}{2}(F_X(Y_3))^2\, dF_X(Y_3) = \frac{1}{3!}(F_X(Y_4))^3$. $Y_3 < Y_4$.

STOP <u>AFTER</u> GET TO Y_{k-1}, SINCE SKIPPING $\int dY_k$. AT THIS POINT, HAVE $\frac{1}{(k-1)!}(F_X(Y_k))^{k-1}$.

- ON THE OTHER SIDE, GET $\frac{1}{(n-k)!}(1 - F_X(Y_k))^{n-k}$ SIMILARLY.
- NEVER DID INTEGRATE $f_X(Y_k)$, SO HAVE THAT ALSO. INCLUDING OVERALL $n!$ ABOVE, GET

$$f_{Y_k}(Y_k) = \frac{1}{(k-1)!}(F_X(Y_k))^{k-1} \frac{1}{(n-k)!}(1 - F_X(Y_k))^{n-k} f_X(Y_k)\, n! = \boxed{\frac{n!}{(k-1)!\,(n-k)!} F_X(Y_k)^{k-1} (1 - F_X(Y_k))^{n-k} f_X(Y_k)}$$

(c) USING MULTINOMIAL FORMULA (FROM LECTURE OR STARK-WOODS p. 24)

$$\cancel{\Delta Y} f_{Y_k}(Y) = \underbrace{\frac{n!}{(k-1)!\,1!\,(n-k)!}}_{(k-1)+1+(n-k)=n} \underbrace{P_r(X < Y)^{k-1}}_{k-1\ X\text{S LESS THAN } Y} \underbrace{P_r[Y < X < Y + \Delta Y]}_{\substack{1\ X \text{ IS BETWEEN } Y \text{ AND } Y+\Delta Y \\ = f_X(Y)\cancel{\Delta Y}}} \underbrace{P_r[X > Y]^{n-k}}_{n-k\ X\text{S GREATER THAN } Y}$$

$$\rightarrow \boxed{f_{Y_k}(Y) = \frac{n!}{(k-1)!\,(n-k)!}(F_X(Y))^{k-1}(1 - F_X(Y))^{n-k} f_X(Y)}$$

AGREES WITH (b)

NOTE IN PARTICULAR:

$\underset{\text{MAXIMUM}}{f_{Y_n}(Y)} = n(F_X(Y))^{n-1} f_X(Y) = \frac{d}{dY}(F_X(Y))^n = \frac{d}{dY}(P_r[X_1 < Y \text{ AND } X_2 < Y \text{ AND} \ldots])$

$\underset{\text{MINIMUM}}{f_{Y_1}(Y)} = n(1 - F_X(Y))^{n-1} f_X(Y) = \frac{d}{dY}[1 - (1 - F_X(Y))^n] = \frac{d}{dY}[1 - P_r[X_1 > Y \text{ AND } X_2 > Y \ldots]]$

REF: A. PAPOULIS, PROB., RANDOM PROC., ..., 3RD ED., 1991, pp. 185–186 (c).
K.S. SHANMUGAN, A.M. BREIPOHL, RANDOM SIGNALS, 1988, pp. 70–72 (a, b).