1a.

1b. $f_x(X) = \frac{d}{dX}F_x(X)$. Area under $f_x(X) = \frac{1}{2}(1)(\frac{1}{2}) + (1)(\frac{1}{4}) + \frac{1}{20}(20-10) = 1$ checks.

1c. **1.** $Pr[x \geq 10] = 1 - F_x(10) = \frac{1}{2}$. **2.** $Pr[x < 5] = F_x(5) = \frac{1}{2}$.

1c. **3.** $Pr[5 < x < 10] = F_x(10) - F_x(5) = 0$. **4.** $Pr[x = 1] = 0$ since $F_x(X)$ continuous.

2a. $Pr[\cup_{i=1}^{3} A_i] = Pr[(A_1 \cup A_2) \cup A_3] = Pr[A_1 \cup A_2] + Pr[A_3] - Pr[(A_1 \cup A_2) \cap A_3]$
$= (Pr[A_1] + Pr[A_2] - Pr[A_1 \cap A_2]) + Pr[A_3] - Pr[(A_1 \cap A_3) \cup (A_2 \cap A_3)]$
$= Pr[A_1] + Pr[A_2] + Pr[A_3] - Pr[A_1 \cap A_2] - (Pr[A_1 \cap A_3] + Pr[A_2 \cap A_3]$
$-Pr[(A_1 \cap A_3) \cap (A_2 \cap A_3)])$. The desired result follows immediately.

2b. $Pr[\cup_{i=1}^{n} A_i] = \binom{n}{1} Pr[A_i] - \binom{n}{2} Pr[A_i \cap A_j] + \binom{n}{3} Pr[A_i \cap A_j \cap A_k] - \ldots$

$= \binom{n}{1}(\frac{1}{n}) - \binom{n}{2}\frac{1}{n(n-1)} + \binom{n}{3}\frac{1}{n(n-1)(n-2)} - \ldots - (-1)^n \binom{n}{n}\frac{1}{n!}$

$= 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots - (-1)^n \frac{1}{n!} \to 1 - e^{-1} = 0.632$ as $n \to \infty$.

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3. $f_{w,y,z}(W,Y,Z) = \int_{-\infty}^{\infty} f_{w,x,y,z}(W,X,Y,Z)dX = \int_1^2 X^2 A(\frac{W}{YZ})^2 \log(\frac{W}{Y})dX$

$= \frac{7}{3}A(\frac{W}{YZ})^2 \log(\frac{W}{Y})$. $f_{x|w,y,z}(X|W,Y,Z) = \frac{f_{w,x,y,z}(W,X,Y,Z)}{f_{w,y,z}(W,Y,Z)} = \begin{cases} \frac{3}{7}X^2, & \text{for } 1 < X < 2; \\ 0, & \text{otherwise} \end{cases}$.

Note that x is independent of $\{w, y, z\}$.

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4. $at^2 + bt + c = 0$ has real roots iff $b^2 - 4ac \geq 0$, so Pr[real roots]=Pr[$b^2 - 4ac \geq 0$]

$= \int_0^\infty dA \int_0^\infty dB \int_0^{\frac{B^2}{4A}} dC f_x(A) f_x(B) f_x(C) = \int_0^\infty dA \int_0^\infty dB\, F_x(\frac{B^2}{4A}) f_x(A) f_x(B)$

since: $F_x(X) = \int_0^X f_x(C)dC$ (recall $x > 0$) and $f_{a,b,c}(A,B,C) = f_x(A)f_x(B)f_x(C)$.

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5. $f_{b,c}(B,C) = 1/(4n^2)$ for $-n < B, C < n$.
Pr[$t^2 + bt + c = 0$ real roots]=Pr[$b^2 \geq 4c$]
$= 1 - Pr[c > \frac{b^2}{4}]$=1-Pr[shaded region]
$= 1 - 2\int_0^{2\sqrt{n}} dB \int_{\frac{B^2}{4}}^{n} dC(\frac{1}{4n^2})$
$= 1 - 2\int_0^{2\sqrt{n}} (n - \frac{B^2}{4})(\frac{1}{4n^2})dB$
$= 1 - \frac{2}{3}\frac{1}{\sqrt{n}} \to 1$ as $n \to \infty$.
So "almost all" quadratics have real roots
if the coefficients are chosen from [-n,n].

6. Let $A = \{(X,Y) : f_m Y < f_x(X)\}$=region of *accepted* x=shaded region below.

$Pr[A] = \int_0^1 dX \int_0^{\frac{f_x(X)}{f_m}} dY\, 1 = \int_0^1 f_x(X)/f_m dX = 1/f_m \leq 1$ since $f_m \geq 1$.

$$f_{x,y|A}(X,Y|A) = \begin{cases} f_{x,y}(X,Y)/Pr[A], & \text{for } (X,Y) \in A \\ 0, & \text{for } (X,Y) \notin A \end{cases} = \begin{cases} f_m, & \text{for } x \in A \\ 0, & \text{for } x \notin A \end{cases}$$

$f_{x|A}(X|A) = \int_0^{\frac{f_x(X)}{f_m}} dY\, f_{x,y|A}(X,Y|A) = \frac{f_x(X)}{f_m} f_m = f_x(X)$, $0 < x < 1$ Q.E.D.
