NOTE: Major point of Problems #1-3 is learning to *read* the problem.

1a. Pr[one BM destroyed]=1-Pr[both AMM miss]=$1-(0.2)^2 = 0.96$.
Pr[all 6 BM destroyed]=$(Pr[\text{BM destroyed}])^6 = (0.96)^6 = 0.783$.
1b. Pr[at least one BM gets through]=1-(the answer to #1a)=0.217.
1c. Pr[exactly one BM gets through]=$\binom{6}{5}(0.96)^5(1-0.96)^1 = 0.195$.

2. Pr[exactly one gets through|target destroyed]=$\frac{\text{Pr[exactly one]}}{\text{Pr[at least one]}} = \frac{0.195(from\,\#1c)}{0.217(from\,\#1b)} = 0.90$

3a. Pr[at least one success in m attempts]=$1-(Pr[no\,success])^m = 1-(1-p^N)^m$.
3b. For *each* receiver, Pr[at least one success in m attempts]=$1-(1-p)^m$.
For *all* receivers, Pr[at least one success in m attempts]=$(1-(1-p)^m)^N$.
NOTE: For p=0.9,N=5,m=2, we get $P(2) = 0.832(from\,(a)) < P_D(2) = 0.951(from\,(b))$.
3c. For N=1 and $p << 1$, using the binomial expansion gives
$1-(1-p)^m = 1-(1-mp+\frac{m(m-1)}{2}p^2+\ldots) \approx mp$ (neglects Pr[2 or more successes]).

NOTE: Major point of Problems #4-#6 is conditional probability.
For Problem #4 count in Ω; for #5-6 use formula and Bayes's rule.

4. Sample space: $\Omega = \{(i,j) : 1 \leq i,j \leq 9\}$. Random variable: $\Sigma = i+j$.
Each point in sample space is equally likely, with $Pr[(i,j)] = \frac{1}{81}$. Events=sets:

$\{\Sigma = 7\} \Leftrightarrow (i,j) \in \{(6,1),(5,2),(4,3),(3,4),(2,5),(1,6)\}$. $\{\Sigma > 10\} : 36\,ways$.
$\{\Sigma > 10\} \cap \{i,j \leq 7\} \Leftrightarrow (i,j) \in \{(4,7),(5,6),(6,5),(7,4),(5,7),(6,6),(7,5),(6,7),(7,6),(7,7)\}$.
$\{\Sigma\, odd\} \cap \{i = 9\} \Leftrightarrow (i,j) \in \{(9,2),(9,4),(9,6),(9,8)\}$. $\{\Sigma\, odd\} : 40\,ways$.

$$Pr[\Sigma = 7|\Sigma\, odd] = \frac{Pr[\{\Sigma=7\}\cap\{\Sigma\, odd\}}{Pr[\Sigma\, odd]} = \frac{Pr[\Sigma=7]}{Pr[\Sigma\, odd]} = \frac{6/81}{40/81} = 0.150. \ (Pr[\Sigma\, odd] \neq \tfrac{1}{2}.)$$

Pr[i or j>7 | $\Sigma > 10] = 1 - Pr[i,j \leq 7|\Sigma > 10] = 1 - \frac{Pr[\{i,j\leq 7\}\cap\{\Sigma>10\}}{Pr[\Sigma>10]} = 1 - \frac{10/81}{36/81} = 0.722$.

$$Pr[\Sigma\, odd|i = 9] = \frac{Pr[\{\Sigma\, odd\}\cap\{i=9\}]}{Pr[i=9]} = \frac{4/81}{9/81} = 0.444.$$

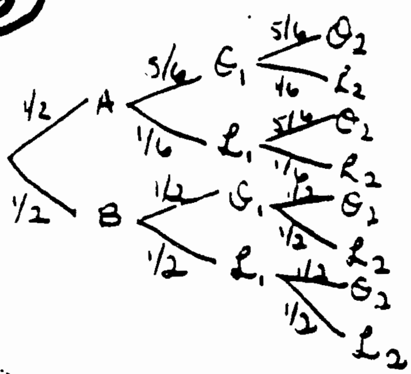
5. We are given the ratios $Pr[x = 3] = 3Pr[x = 1]; Pr[x = 2] = 2Pr[x = 1]$.
Then $Pr[x = 1] + Pr[x = 2] + Pr[x = 3] = 1 \rightarrow Pr[x = X] = X/6, X = 1,2,3$.

$$Pr[x = 1|y = 1] = \frac{Pr[y=1|x=1]Pr[x=1]}{\sum_{n=1}^{n=3} Pr[y=1|x=n]Pr[x=n]} = \frac{(1-\alpha)\frac{1}{6}}{(1-\alpha)\frac{1}{6}+\frac{\beta}{2}\frac{2}{6}+\frac{\gamma}{2}\frac{3}{6}} = \frac{1-\alpha}{1-\alpha+\beta+3\gamma/2}.$$

This is an application of *Bayes's rule*; note the (very common) form $\frac{A}{A+B+C}$.

Let events be: A: choose die A
B: choose die B
O_n: olive face on throw n
L_n: lavender face on throw n

(6)



a) $\text{Prob}(O_n) = P(O_n A) + P(O_n B)$
$= P(O_n|A)P(A) + P(O_n|B)P(B)$
$= (5/6)(1/2) + (1/2)(1/2) = 2/3$

b) $\text{Prob}(O_n \text{ and } O_{n+1}) = P[(O_n \text{ and } O_{n+1})A] + P[(O_n \text{ and } O_{n+1})B]$
$= P[(O_n \text{ and } O_{n+1})|A]P(A) + P[(O_n \text{ and } O_{n+1})|B]P(B)$
$= (5/6)(5/6)(1/2) + (1/2)(1/2)(1/2) = 17/36$

c) $$\text{Prob}(O_{n+1}|O_1, O_2, \dots O_n) = \frac{\text{Prob}(O_1, O_2, \dots O_n, O_{n+1})}{\text{Prob}(O_1, O_2, \dots O_n)}$$

Extrapolating from part b), this is $\dfrac{(5/6)^{n+1}(1/2) + (1/2)^{n+1}(1/2)}{(5/6)^n(1/2) + (1/2)^n(1/2)}$

$$= \boxed{\frac{(5/6)^{n+1} + (1/2)^{n+1}}{(5/6)^n + (1/2)^n}}$$

Dividing top and bottom by $(5/6)^n$ gives $\dfrac{(5/6) + (1/2)(3/5)^n}{1 + (3/5)^n}$

As n becomes large, $(3/5)^n \to 0$ and the fraction approaches $5/6$, indicating that die A is being used.

(7)

7 a. We consider here an experiment which consists of drawing k balls from a box containing $n + m$ balls. Since k balls can be selected from the $n + m$ in $\binom{n+m}{k}$ different ways, there are $\binom{n+m}{k}$ equally likely possible outcomes to our experiment. The favorable ones are those in which exactly r of the k balls drawn are white, and consequently, $k - r$ of them black. It is clear that this is possible only when $r \leq k$, $r \leq n$, and $k - r \leq m$; when these conditions do not all hold, there are no favorable outcomes at all and the desired probability is zero. Assuming that the indicated inequalities are satisfied, we obtain all the favorable outcomes by combining each of the $\binom{n}{r}$ ways of drawing r balls from the n white balls with the $\binom{m}{k-r}$ ways of drawing $k - r$ balls from the m black balls. Thus the number of favorable outcomes is $\binom{n}{r}\binom{m}{k-r}$, and the required probability is $\boxed{\binom{n}{r}\binom{m}{k-r}\Big/\binom{n+m}{k}}$.

HYPERGEOMETRIC FORMULA, DIRECTLY FROM LECTURE.

7 b. Let $k \leq n$ and $k \leq m$. When k balls are drawn from a box containing n white balls and m black ones, the number of white balls drawn can be $0, 1, 2, \dots,$ or k. According to part a, the probability that this number will be 0 is $\binom{n}{0}\binom{m}{k}\Big/\binom{n+m}{k}$, the probability that it will be 1 is $\binom{n}{1}\binom{m}{k-1}\Big/\binom{n+m}{k}$. The probability is $\binom{n}{2}\binom{m}{k-2}\Big/\binom{n+m}{k}$ that it will be $2, \dots,$ and the probability is $\binom{n}{k}\binom{m}{0}\Big/\binom{n+m}{k}$ that it will be k. But the sum of these probabilities is 1, since exactly one of the above events must occur. Thus

$$\frac{\binom{n}{0}\binom{m}{k}}{\binom{n+m}{k}} + \frac{\binom{n}{1}\binom{m}{k-1}}{\binom{n+m}{k}} + \frac{\binom{n}{2}\binom{m}{k-2}}{\binom{n+m}{k}} + \cdots + \frac{\binom{n}{k}\binom{m}{0}}{\binom{n+m}{k}} = 1,$$

and consequently

$$\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \binom{n}{2}\binom{m}{k-2} + \cdots + \binom{n}{k}\binom{m}{0} = \binom{n+m}{k}. \blacksquare$$

THERE ARE MANY WEIRD IDENTITIES LIKE THIS. MOST CAN BE INTERPRETED IN TERMS OF COUNTING.