1. Let $b_i, i = 1 \ldots 6$ denote the i^{th} ball in the urn, and
   $b_i b_j$ denote that the i^{th} THEN j^{th} balls are taken (note that order matters).

1a. **without replacement:** Ω has $(6)(6-1) = 30$ elements, which are:
$\Omega = \{b_1b_2, b_1b_3, b_1b_4, b_1b_5, b_1b_6, b_2b_1, b_2b_3, b_2b_4, b_2b_5, b_2b_6 \ldots b_5b_6\}$.

1b. **with replacement:** Ω has $(6)(6) = 36$ elements, which are:
$\Omega = \{b_1b_1, b_1b_2, b_1b_3, b_1b_4, b_1b_5, b_1b_6, b_2b_1, b_2b_2, b_2b_3, b_2b_4 \ldots b_6b_6\}$.

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2. Let H=height of man in inches and W=height of woman in inches.

2a. $\Omega = \{(H, W) : H > 0 \, and \, W > 0\} = (\mathcal{R}^+)^2$. **(b):** $E = \{(H, W) : 0 < H < W\}$.

3a. $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset \rightarrow Pr[A] = Pr[A \cup \emptyset] = Pr[A] + Pr[\emptyset] \rightarrow Pr[\emptyset] = 0$.

3b. $(EF) \cup (EF') = E$ and $(EF) \cap (EF') = \emptyset \rightarrow Pr[E] = Pr[EF] + Pr[EF']$.

3c. $E \cup E' = \Omega$ and $E \cap E' = \emptyset \rightarrow Pr[E] + Pr[E'] = Pr[\Omega] = 1 \rightarrow Pr[E] = 1 - Pr[E']$.

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4. $(EF') \cup (FE') = E \oplus F$ and $(EF') \cap (FE') = \emptyset \rightarrow Pr[E \oplus F] = Pr[EF'] + Pr[FE']$.

5. $\#3b \rightarrow Pr[EF'] = Pr[E] - Pr[EF]$ and $Pr[FE'] = Pr[F] - Pr[FE]$.
   $\#4 \rightarrow Pr[E \oplus F] = Pr[E] + Pr[F] - 2Pr[EF]$ QED. Note $Pr[EF] = Pr[FE]$.

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6. See overleaf.

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7. Let $B = \cap_{n=1}^{\infty} A_n = \cap_{n=1}^{\infty} (0, 1/n)$. $\quad x \in B \Leftrightarrow x \in A_n \Leftrightarrow 0 < x < 1/n$ for all n.
   But for *any* $x > 0, \exists N \, s.t. \, 1/N < x \rightarrow x \notin A_N \rightarrow x \notin B$. Hence $B = \emptyset$.

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8. Let $\mathcal{Z} = \{integers\}$ and $P_n = \{$polynomials of degree n with integer coefficients$\}$.
   P_n is 1-1 with $\mathcal{Z}^{n+1}$ since $a_0 + a_1 z + \ldots + a_n z^n \in P_n \leftrightarrow (a_0, a_1 \ldots a_n) \in \mathcal{Z}^{n+1}$.
   $\mathcal{A} = \{$algebraic irrationals$\}$ 1-1 with a *subset* of $\cup_{n=1}^{\infty} P_n$, since every
   $A \in \mathcal{A}$ is the root of a polynomial with integer coefficients (some duplication).
   $\cup_{n=1}^{\infty} P_n$=countable union of countable sets=countable$\rightarrow \mathcal{A}$ *at most* countable.
   $\{2^{1/n}, n = 2, 3 \ldots\} \subset \mathcal{A} \rightarrow \mathcal{A}$ *at least* countably infinite $\rightarrow \mathcal{A}$ is countably infinite.

6. Note that there are L^K different mappings $f : \{1, 2 \dots K\} \to \{1, 2 \dots L\}$.
Try interpreting the following results in terms of this result.

6a. Typical member of A is specified by $\{f(0) = i, f(1) = j\}$ where $i, j \in \mathcal{Z}^+$.
A is 1-1 with $(\mathcal{Z}^+)^2$ since $\{f(0), f(1)\} \leftrightarrow (i, j)$. COUNTABLE.

6b. B_n is 1-1 with $(\mathcal{Z}^+)^n$ since $\{f(1) \dots f(n)\} \leftrightarrow (i_1, \dots i_n)$. COUNTABLE.

6c. C is a countable union of countable sets B_n from #6b, so C is COUNTABLE.

6d. $E \subset D$ and D is uncountable from #6e below, so D is UNCOUNTABLE.

6e. Typical member of E is specified by $f(1) = 0, f(2) = 1, f(3) = 1, f(4) = 0 \dots$
E is 1-1 with $[0, 1)$ since $\{f(1), f(2) \dots\} \leftrightarrow (0.x_1x_2 \dots) \in [0, 1)$ where x_i=0,1.
Since $[0, 1)$ is uncountable, E is UNCOUNTABLE. (D, E are only uncountables).

6f. $F = \cup_{N=1}^{\infty} F_N$ where $F_N = \{f : f(n) = 0 \, for \, n > N\}$ has 2^N elements.
F is a countable union of *finite* sets F_N, so F is COUNTABLE.
NOTE: F does *not* include an "F_∞" which would have "2^∞" elements.

6g. $G = \cup_{N=1}^{\infty} G_N$ where $G_N = \{f : f(n) = 1 \, for \, n > N\} = B_N$ from #6b.
G is a countable union of *countable* sets G_N, so G is COUNTABLE.

6h. Let $H_{i,j}$ be the set of functions $f(n)$ that are eventually j for $n > i$.
NOTE: The "eventual constant" must be an integer since $f : Z^+ \to Z^+$.
$H = \cup_{i=1}^{\infty} \cup_{j=1}^{\infty} H_{i,j}$. Note that $H_{N,1} = G_N$ from #6g.
H is a countable double union of countable sets, so H is COUNTABLE.

6i. $I = \{\{i, j\} : i \neq j \, and \, i, j \in \mathcal{Z}^+\} \leftrightarrow (\mathcal{Z}^+)^2$, excluding the diagonal lattice points.
Typical members of I:$\{1, 2\}, \{3, 5\}, \{4, 9\} \dots$ I is COUNTABLE.

6j. $J = \cup_{n=1}^{\infty} J_n$ where $J_n = \{\{i_1, i_2 \dots i_n\} : i_1 \neq i_2 \neq \dots i_n \, and \, i_1 \dots i_n \in \mathcal{Z}^+\}$.
Note $J_2 = I$ from #6i. $J_n \leftrightarrow (\mathcal{Z}^+)^n$, again excluding diagonal lattice points.
J is a countable union of countable sets, so J is COUNTABLE.
