Given: Random process $x(t) = A\cos(\omega t + \theta)$ where ω is a known constant.
Random: A and θ are *independent random variables.* $f_\theta(\Theta) = \frac{1}{2\pi}, 0 < \Theta < 2\pi$.
Goal: Compute the mean $E[x(t)]$ and covariance $K_x(t,s)$ functions of $x(t)$.

Mean: $E[x(t)] = E[A]E[\cos(\omega t + \theta)] = E[A]\int_0^{2\pi} \frac{1}{2\pi}\cos(\omega t + \Theta)d\Theta = 0.$

Covar-iance: $K_x(t,s) = R_x(t,s) = E[x(t)x(s)] = E[A^2]E[\cos(\omega t + \theta)\cos(\omega s + \theta)]$
$= E[A^2]\frac{1}{2}E[\cos(\omega(t+s) + 2\theta) + \cos(\omega(t-s))] = \frac{1}{2}E[A^2]\cos(\omega(t-s)).$

Note: (1) $x(t)$ WSS; (2) Need only $E[A^2]$, *not* $f_a(A)$; (3) Can have $E[A] \neq 0$.

Now: Let $A > 0$ be a known constant. Compute $f_{x(t)}(X)$ and $f_{x(t)|x(s)}(X_t|X_s)$.
Note: Sample functions are sinusoids with frequency ω and amplitude A.
Different sample points→different phases→different sample functions.
EX: $x(t)$ is ideal oscillator with known amp. and freq. but random phase.

$f_{x(t)}(X)$: t is fixed→this is a *derived distribution* problem from θ to $x(\theta)$.
Jaco-bian: $f_x(X) = \sum_1^2 (1/|\frac{dx}{d\theta}|) f_\theta(\Theta_i)|_{\Theta_i = x^{-1}(X)}$, where $\frac{dx}{d\theta} = \frac{d}{d\theta}A\cos(\omega t + \theta)$
$= -A\sin(\omega t + \theta) = \pm A\sqrt{1 - \cos^2(\omega t + \theta)} = \pm\sqrt{A^2 - X^2}.$

$f_x(X) = \frac{1}{\sqrt{A^2 - X^2}}\frac{1}{2\pi} + \frac{1}{\sqrt{A^2 - X^2}}\frac{1}{2\pi} = 1/(\pi\sqrt{A^2 - X^2}), |X| < A.$

Note: Integrates to one; diverges at $X = \pm A$ (most likely values of $x(t)$).
Note: $\exists$ 2 solns Θ_1, Θ_2 to $x(\theta) = X \to \theta = x^{-1}(X)$: x sweeps $[-A, A]$ *twice.*

Next: $f_{x(t)|x(s)}(X_t|X_s) = \frac{1}{2}\delta(X_t - X_1) + \frac{1}{2}\delta(X_t - X_2)$, where X_1, X_2 are
$x(s) = A\cos(\omega s + \theta) = X_s \to \theta = \cos^{-1}\frac{X_s}{A} - \omega s \to 2$ values of θ:
$\to x(t) = A\cos(\omega t + \theta_1) = X_1$ or $x(t) = A\cos(\omega t + \theta_2) = X_2$.
Note: Each of X_1 and X_2 is equally likely with probability 1/2, hence $\delta(\cdot)$.
Note: 1^{st}-order stationary⇔id; *not* independent or 2^{nd}-order stationary.

Next: $f_{x(t),x(s)}(X_t, X_s) = f_{x(t)|x(s)}(X_t|X_s)f_{x(s)}(X_s)$. All of these now known.

$$f_{x(t),x(s)}(X_t, X_s) = \frac{1}{2\pi}\frac{1}{\sqrt{A^2 - X_s^2}}(\delta(X_t - X_1) + \delta(X_t - X_2)), |X_s| < A$$

Note: Once have $f_{x(t),x(s)}(X_t, X_s)$, can get any 2^{nd}-order statistic or Pr[event].
