Given: \( x(t) \) is a real-valued 0-mean WSS random process (RP).
With: Autocorrelation \( R_x(\tau) = E[x(t)x(t+\tau)] \) for any time \( t \),
Note: \( x(t) \) 0-mean → \( R_x(\tau) = K_x(\tau) \)=covariance func. & lag \( \tau \).
WSS: \( E[x(t)x(s)] = R_x(t-s) \); not function of \( t \) and \( s \) separately.

DEF: Power spectral density (PSD) \( S_x(\omega) \) is defined as:
\[
S_x(\omega) = \mathcal{F}\{R_x(\tau)\} = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega \tau}d\tau = 2\int_{0}^{\infty} R_x(\tau)\cos(\omega \tau)d\tau.
\]
\[
R_x(\tau) = \mathcal{F}^{-1}\{S_x(\omega)\} = \int_{-\infty}^{\infty} S_x(\omega)e^{j\omega \tau}d\omega = \int_{0}^{\infty} S_x(\omega)\cos(\omega \tau)\frac{d\omega}{\pi}.
\]
Assume: \( x(t) \) is 2nd-order process: \( E[x(t)^2] < \infty \) (except: white).

Properties of Power Spectral Density
1. \( x(t) \) real → \( R_x(\tau) = R_x(-\tau) \) → \( S_x(\omega) = S_x(-\omega) \) is real:
\[
\mathcal{F}\{\text{real, even function}\} = \text{real, even function} \rightarrow \text{cosine xform}.
\]
2. \( R_x(\tau) \) is positive semidefinite \( \Leftrightarrow S_x(\omega) \geq 0 \):
\[
\sigma_y^2 = \int \int f(t)x(t)dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)R_x(t-s)f^*(s)ds dt \geq 0.
\]

Proof: \( \Rightarrow \): Suppose \( \exists \omega_o \) so that \( S_x(\omega_o) < 0 \). Let \( f(t) = e^{j\omega_0 t} \).
Then: \( \sigma_y^2 = \int \int e^{j\omega_0 t}R_x(t-s)e^{-j\omega_0 s}dt ds = S_x(\omega_o) \cdot \infty < 0 \).

Proof: \( \Leftarrow \): For any real \( f(t) \), write \( f(t) = \int F(\omega)e^{j\omega t}d\omega \).
Then: \( \sigma_y^2 = \int |F(\omega)|^2S_x(\omega)d\omega \geq 0 \) using Parseval twice.

Note: Much simpler in the frequency domain! (just nonnegative)
4. Average power = \( E[x(t)^2] = \sigma_x^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)d\omega \).

Note: “Power” is the tendency of random \( |x(t)| \) to be large:
Larger variance → broader pdf → RV tends to be larger.
EX: Let \( v(t) \)=random voltage across a resistor \( R = 1\Omega \).
Average power = \( E[v(t)^2] \); large despite \( E[v(t)] = 0 \).
1. LTI system; impulse response \( h(t) : \delta(t) \rightarrow |h(t)| \rightarrow h(t) \)

2. LTI system has transfer function \( H(\omega) = \mathcal{F}\{h(t)\} \):
   \[
   \cos(\omega t) \rightarrow |h(t)| \rightarrow |H(\omega)| \cos(\omega t + \text{ARG}[H(\omega)])
   \]

3. WSS random processes: \( x(t) \rightarrow |h(t)| \rightarrow y(t) \)

4. IMPORTANT FORMULA: \( S_y(\omega) = |H(\omega)|^2 S_x(\omega) \).

From: Take \( \mathcal{F} \) of \( R_y(\tau) = \int \int h(u)h(v)R_x(\tau - u + v)du \, dv \).

Note: Compare to random vectors: \( y = Ax \rightarrow K_y = AK_xA^T \).

EX: \( x(t) \rightarrow |dy/dt + ay(t) = x(t)| \rightarrow y(t), \quad a > 0 \) so stable.

\( x(t) \) is a 0-mean uncorrelated WSS RP. What is \( S_y(\omega) \)?

1. 0-mean WSS uncorrelated \( \rightarrow R_x(\tau) = \delta(\tau) \rightarrow S_x(\omega) = 1. \) (Assume WLOG that the area under impulse is unity.)

2a. Impulse response: \( h(t) = e^{-at} \) for \( t \geq 0 \); 0 otherwise.

2b. Transfer function: \( H(\omega) = \mathcal{F}\{h(t)\} = 1/(j\omega + a) \).

3. \( S_y(\omega) = |1/(j\omega + a)|^2 \cdot 1 = 1/(\omega^2 + a^2) \).

4. \( R_y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} e^{j\omega \tau} d\omega = \frac{1}{2a} e^{-a|\tau|} \).

5. \( \sigma_y^2(t) = E[y(t)^2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} d\omega = R_y(0) = \frac{1}{2a} \).

6. \( x(t) \) Gaussian \( \rightarrow y(t) \) Gaussian \( \rightarrow f_{y(t)}(Y) \sim N(0, \frac{1}{2a}) \).

7. See p. 487 of Stark and Woods for more details.

EX: \( x(t) \) Gaussian; \( a = 5 \).

\( Pr[1 < x(5) < 2 & 3 < x(6) < 4] = ? \)

Soln: \( Pr = \int_1^2 \int_3^4 \frac{1}{2\pi} \frac{1}{\sqrt{\det[K]}} e^{-\frac{1}{2} [X_5, X_6]K^{-1}[X_5, X_6]' dX_6 dX_5 \}

where: \( K = \frac{1}{10} \begin{bmatrix} 1 & e^{-5} \\ e^{-5} & 1 \end{bmatrix} \rightarrow \det[K] = 0.01(1 - e^{-10}) \).
**Def:** A 0-mean WSS RP $x(t)$ is a white process if $S_x(\omega) = \sigma^2$ for some positive constant $\sigma^2 > 0$.

**Comments on White Processes:**

1. All frequencies $\omega$ equally represented $\rightarrow$ ”white” process: Red+orange+yellow+green+blue+violet+others=white if all colors (even not listed) present in equal strengths.

2. $R_x(\tau) = \sigma^2 \delta(\tau)$; $x(t)$ is an uncorrelated process.

3. Power $= R_x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma^2 d\omega \to \infty!$ (impulse at $\tau = 0$).
   a. Infinite power $\rightarrow$ white process cannot exist physically!
   b. NOT a 2nd-order process. Still: often used in models.

4. Implications of continuous-time uncorrelated RP:
   a. Knowledge of $x(7)$ does not help you predict $x(7.000001)$;
   b. Typical sample function (realization) of a white RP: $\sim$ scatter plot (dust sprinkled on figure with $t$ axis).
   c. Takes infinite power to be able to move from $x(7)$ to different value $x(7.000001)$ in almost-zero time.

**So What Good are White RPs If They Don’t Exist?**

1a. In modelling: Usually pass white $x(t)$ through LTI system.
1b. Real-life systems have finite bandwidth: $\lim_{\omega \to \infty} |H(\omega)| = 0$.
1c. So $S_x(\omega)$ doesn’t matter for large $\omega$: gets filtered anyway.
   White input and bandlimited white input $\rightarrow$ same output.

2. A 2nd-order 0-mean WSS RP $x(t)$ can be modelled as:
   white RP $\rightarrow$ $|H(\omega) = \sqrt{S_x(\omega)}| \to x(t)$. OR: $H(\omega)$ causal.

$$H(s) = \prod \frac{(s+z_i)(s+z_i^*)}{(s+p_i)(s+p_i^*)} \rightarrow S_x(\omega) = \prod \frac{(\omega^2+z_i^2)(\omega^2+z_i^{*2})}{(\omega^2+p_i^2)(\omega^2+p_i^{*2})} = \prod \frac{|\omega^2+z_i^2|^2}{|\omega^2+p_i^2|^2}.$$  

**using:** $(j\omega+z)(-j\omega+z)(j\omega+z^*)(-j\omega+z^*) = (\omega^2+z^2)(\omega^2+z^{*2})$.  

$$\prod \frac{(\omega^2+z_i^2)(\omega^2+z_i^{*2})}{(\omega^2+p_i^2)(\omega^2+p_i^{*2})}.$$
1. \( x(t) \rightarrow |H(\omega)| \rightarrow y(t) \), \( H(\omega) = \begin{cases} 1, & \text{if } 3 \leq |\omega| \leq 3.001; \\ 0, & \text{otherwise.} \end{cases} \)

2. Then \( S_y(\omega) = \begin{cases} S_x(\omega) \approx S_x(3), & \text{if } 3 \leq |\omega| \leq 3.001; \\ 0, & \text{otherwise.} \end{cases} \)

3. Then the average power \( E[y(t)^2] \) in output \( y(t) \) is:

\[
E[y(t)^2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega)d\omega = \frac{2}{2\pi} \int_{3}^{3.001} S_x(\omega)d\omega \approx \frac{0.001}{\pi} S_x(3).
\]

4. **Interpretation:** \( S_x(3) \) is the average power per unit bilateral bandwidth in \( x(t) \) at \( \omega = 3 \):
   
   a. **Bilateral:** components at both \( \omega = 3 \) and \( \omega = -3 \);
   b. \( \frac{2\Delta}{2\pi} S_x(\omega_o) \) is the average power in random process \( x(t) \) in the frequency band of width \( \Delta \): \( \omega_o \leq \omega \leq \omega_o + \Delta \).
   c. **Units:** \( x(t) \) volts \( \rightarrow \frac{1}{2\pi} S_x(\omega) \frac{\text{volts}^2}{\text{rad/sec}}; \ S_x(f) \frac{\text{volts}^2}{\text{Hertz}}. \)

5. \( S_x(\omega) \) must be multiplied by frequency to get power in a frequency band; it is a power spectral **density**:
   
   a. \( \bullet \ S_x(\omega_o)\frac{2\Delta}{2\pi} = \text{Power in } [\omega_o \leq \omega \leq \omega_o + \Delta]. \)
   b. \( \bullet \ S_x(f_o)2\Delta = \text{Power in } [f_o \leq f \leq f_o + \Delta]. \)
   c. **Compare:** \( f_x(X)\Delta = Pr[X \leq x \leq X + \Delta]. \)

6. This interpretation makes the following evident:
   
   a. \( S_x(\omega) \geq 0 \): otherwise power in some frequency band would be negative! **Compare to:** \( f_x(X) \geq 0. \)
   b. Total average power=\( E[x(t)^2] = \int_{-\infty}^{\infty} S_x(f)df. \)
   
   **Compare to:** Total probability=\( \int_{-\infty}^{\infty} f_x(X)dX = 1. \)
1. Recall we can *decorrelate* a random N-vector \( x \) as follows:
   a. Covariance matrix \( K_x \) has \( N \) eigenvalues \( \lambda_i \) and eigenvectors \( \phi_i, \) \( i = 1 \ldots N \), where \( K_x \phi_i = \lambda_i \phi_i \).
   b. Let \( A = [\phi_1 \ldots \phi_N]^T \) (don’t forget transpose!) and \( y = Ax \). Then: \( y_i = \phi_i^T x = \phi_i \cdot x = \sum_{j=1}^{N} (\phi_i)_j x_j \).
   c. Then \( K_y = AK_xA^T = \text{DIAG}[\lambda_1 \ldots \lambda_N] \) and then \( E[y_i y_j] = \lambda_i \delta(i-j) \) \( \rightarrow \{y_i\} \) have been decorrelated.
   d. \( y = Ax \rightarrow x = A^T y \rightarrow x = \sum_{i=1}^{N} y_i \phi_i \): \( x \) = sum of uncorrelated RVs \( \times \) eigenvectors.

2. Now try this for 2nd-order 0-mean WSS processes:
   a. Eigenfunctions of LTI systems: \( \phi(t) = e^{j\omega t} \): Means \( e^{j\omega t} \rightarrow |H(\omega)| \rightarrow H(\omega)e^{j\omega t} = |H(\omega)|e^{(j\omega t + \text{ARG}[H(\omega)])} \).

3. \( x(t) \) is a real-valued 2nd-order 0-mean WSS process.
   a. Define RVs \( X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \), for all \( \omega \).
   b. Then \( \{X(\omega)\} \) are uncorrelated random variables:
   \[ E[X(\omega_1)X^*(\omega_2)] = 2\pi S_x(\omega_1)\delta(\omega_1 - \omega_2). \]

4. Spectral interpretation of 2nd-order WSS RPs:
   \( x(t) = \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} \frac{d\omega}{2\pi} : \int \) uncorrelated RVs \( \times \) eigenfunctions.

5. We have for *finite but large* \( T \) and interval \( [-T/2, T/2] \):
   \[ \textbf{K-L:} \ x(t) = \sum_{n=-\infty}^{\infty} x_n \frac{1}{\sqrt{T}} e^{j2\pi nt/T}, \ |t| \leq T/2 \rightarrow \infty, \]
   where: \( x_n = \int_{-T/2}^{T/2} x(t) \frac{1}{\sqrt{T}} e^{-j2\pi nt/T} dt \) for integers \( n \).
   Then: \( E[x_i x_j] = S_x(2\pi i/T)\delta(i-j) \rightarrow \{x_n\} \) uncorrelated.
   \[ \textbf{DEF:} \ \text{This is Karhunen-Loeve expansion for WSS processes.} \]
   \[ \textbf{Works:} \ S_x(\omega) \approx 0 \ for \ |\omega| > B \rightarrow \text{need} \ TB >> 1. \]
1. \( E[X(\omega_1)X^*(\omega_2)] = E[\int x(t)e^{-j\omega_1 t}dt \int x(s)e^{j\omega_2 s}ds] = \int \int E[x(t)x(s)]e^{-j(\omega_1 t-\omega_2 s)}dt ds. \) \( E[x(t)x(s)] = R_x(t-s). \)

2. Change variables: \( t, s \rightarrow \tau = t - s, z = t + s : |J| = 2. \)

\[
E[X(\omega_1)X^*(\omega_2)] = \int \int R_x(\tau)e^{-j[\omega_1(\tau+z)+\omega_2(\tau-z)]/2} \frac{d\tau dz}{2}
\]

\[
= \int R_x(\tau)e^{-j\frac{\omega_1+\omega_2}{2}\tau} d\tau \int e^{-j\frac{\omega_1-\omega_2}{2}z} \left( \frac{dz}{2} \right) [\mathcal{F}\{1\} = 2\pi \delta(\omega)]
\]

\[
= 2\pi S_x \left( \frac{\omega_1+\omega_2}{2} \right) \delta(\omega_1 - \omega_2) = 2\pi S_x(\omega_1)\delta(\omega_1 - \omega_2). \quad \text{QED.}
\]

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**Gaussian RPs and The Distribution of** \( X(\omega) \)

1. Let \( x(t) \) be Gaussian RP \( \rightarrow X(\omega) \) Gaussian RVs:
   a. \( \text{Re}[X(\omega)] \) and \( \text{Im}[X(\omega)] \) are Gaussian RVs;
   b. \( |X(\omega)| \) Rayleigh RVs; \( \text{ARG}[X(\omega)] \) uniform RVs.

   Rayleigh pdf: \( f_z(Z) = \frac{Z}{\sigma^2} e^{-Z^2/(2\sigma^2)}, Z \geq 0. \) p. 138.

2a. **Physical** Let \( H(\omega) = \delta(\omega - \omega_o) + \delta(\omega + \omega_o): \)

   **interpretation:** \( H(\omega) \) passes ONLY frequency \( \omega_o. \)

   b. \( x(t) \rightarrow |H(\omega)| \rightarrow y(t); \) \( H(\omega) \) narrowband.

3. All possible sample functions of RP \( x(t) \) are filtered.
   a. \( y(t) = A \cos(\omega_o t) + B \sin(\omega_o t) \) for RVs \( A, B. \)
   b. Jointly Gaussian RVs \( A, B \sim N(0, S_x(\omega_o)\pi \infty). \)

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**REF:** Papoulis, 3\(^{rd}\) ed. (1991), pp. 416-418.
Let: \(x(t)\) and \(y(t)\) be real-valued 0-mean jointly WSS RPs.

with: Cross-correlation \(R_{xy}(\tau) = E[x(t)y(t - \tau)]\) for any \(t\).

Means: \(E[x(t)y(s)] = R_{xy}(t - s)\); jointly WSS \(\rightarrow\) not \(t, s\) separately.

Def: The Cross-Spectral Density \(S_{xy}(\omega)\) is defined as:

\[
S_{xy}(\omega) = \mathcal{F}\{R_{xy}(\tau)\}; \quad R_{xy}(\tau) = \mathcal{F}^{-1}\{S_{xy}(\omega)\}.
\]

Properties of The Cross-Spectral Density

1. \(R_{yx}(\tau) = R_{xy}(-\tau) \rightarrow S_{yx}(\omega) = S_{xy}(-\omega) = S_{xy}^*(\omega)\).
2. \(S_{xy}(\omega)\), unlike \(S_x(\omega)\), is not a real or even function.
3. WSS random processes: \(x(t) \rightarrow |H(\omega)| \rightarrow y(t)\)

FORMULA: \(S_{yx}(\omega) = H(\omega)S_x(\omega)\) (note \(S_{yx}\), not \(S_{xy}\)):

Take \(\mathcal{F}\) of \(R_{yx}(\tau) = \int h(u)R_x(\tau - u)du\).

4. From \#1 and \#3 we have \(S_{xy}(\omega) = H^*(\omega)S_x(\omega)\).
5. Exchange \(x(t), y(t)\) and replace \(H(\omega)\) with \(\frac{1}{H(\omega)}\):

\[
S_{yx}(\omega) = \frac{1}{H^*(\omega)}S_y(\omega) \rightarrow S_y(\omega) = |H(\omega)|^2S_x(\omega)!
\]

Example of Cross-Spectral Density

Let \(x(t) \rightarrow |F(\omega)| \rightarrow y(t)\) and \(x(t) \rightarrow |G(\omega)| \rightarrow z(t)\)

Compute the cross-spectral density \(S_{yz}(\omega)\). Solution:

1. \(x(t), y(t), z(t)\) are real-valued 0-mean jointly WSS RPs.
2. \(y(t) = \int f(u)x(t - u)du\) and \(z(s) = \int g(v)x(s - v)dv\) →
3. \(E[y(t)z(s)] = \int \int f(u)g(v)E[x(t - u)x(s - v)]du dv \rightarrow R_{yz}(t - s) = \int \int f(u)g(v)R_x(t - s - u + v)du dv \rightarrow S_{yz}(\omega) = F(\omega)G^*(\omega)S_x(\omega)\). Neat formula! (I think so)
4. Passbands of \(F(\omega)\) and \(G(\omega)\) don’t overlap \(\rightarrow R_{yz}(\tau) = 0\). \(y(t)\) and \(z(t)\) (different frequency components of \(x(t)\)) are uncorrelated \(\rightarrow\) spectral interpretation of WSS RPs.
Infinite Smoothing Filter for Signal in Noise:

1. \( y(t), x(t), v(t) \) are 2nd-order 0-mean jointly WSS RPs.
   a. Observe \( y(t) = x(t) + v(t) \) where \( E[x(t)v(s)] = 0 \).
   b. \( x(t) = \text{signal}, v(t) = \text{noise}, y(t) = \text{noisy data} \).

2. We have the following (cross)covariance functions:
   a. \( R_{xy} = E[x(t)y(s)] = E[x(t)(x(s) + v(s))] = R_x \).
   b. \( R_y = E[(x(t) + v(t))(x(s) + v(s))] = R_x + R_v \).

3. GOAL: Compute LLSE of \( x(t) \) from \( \{y(s), -\infty < s < \infty\} \).
   NOTE: \( x(t), v(t) \) jointly Gaussian RPs \( \rightarrow \hat{x}_{\text{LLSE}} = \hat{x}_{\text{LS}} \).

4. LEMMA: 0-mean uncorrelated \( \{x(i)\} \) and \( \{v(i)\} \).
   Observe \( y(n) = x(n) + v(n) \) where \( E[x(i)v(j)] = 0 \).
   Then LLSE of \( x(i) \) from \( \{y(j), -\infty < j < \infty\} \) is
   \[
   \hat{x} = K_{xy} K_y^{-1} y = K_x(K_x + K_v)^{-1} y
   = \text{DIAG}[\sigma^2_{x(i)}] \text{DIAG}[\sigma^2_{x(i)} + \sigma^2_{v(i)}]^{-1} y
   \]
   \( \rightarrow \hat{x}(i) = \frac{\sigma^2_{x(i)}}{\sigma^2_{x(i)} + \sigma^2_{v(i)}} y(i) \) where \( y = \text{vector of} \{y(j)\} \).
   Problem decouples since the \( \{y(j)\} \) are uncorrelated.

5. Write \( x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \frac{d\omega}{2\pi} \); write \( y(t), v(t) \) similarly.
   Apply LEMMA to RVs \( X(\omega), Y(\omega), V(\omega) \). Solution:
   \[
   y(t) \rightarrow \left| \frac{S_x(\omega)}{S_x(\omega) + S_v(\omega)} \right| \rightarrow \hat{x}(t).
   \]

6. Comments:
   a. More insightful than Recitation derivation!
   b. Shows significance of decorrelation=prewhitening: Eliminates need to compute \( K_y^{-1} \) (saves much work).
   c. See Stark and Woods p. 553 for more details.