Given: $x(t)$ is a real-valued 0-mean WSS random process (RP).
With: Autocorrelation $R_x(\tau) = E[x(t)x(t \pm \tau)]$ for any *time* t,
Note: $x(t)$ 0-mean$\rightarrow R_x(\tau) = K_x(\tau)$=covariance func. & *lag* τ.
WSS: $E[x(t)x(s)] = R_x(t-s)$; not function of t and s separately.

DEF: **Power spectral density** (PSD) $S_x(\omega)$ is defined as:

$$S_x(\omega) = \mathcal{F}\{R_x(\tau)\} = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau}d\tau = 2\int_0^{\infty} R_x(\tau)\cos(\omega\tau)d\tau.$$

$$R_x(\tau) = \mathcal{F}^{-1}\{S_x(\omega)\} = \int_{-\infty}^{\infty} S_x(\omega)e^{j\omega\tau}\frac{d\omega}{2\pi} = \int_0^{\infty} S_x(\omega)\cos(\omega\tau)\frac{d\omega}{\pi}.$$

Assume: $x(t)$ is *2nd-order process*: $E[x(t)^2] < \infty$ (except: white).

Properties of Power Spectral Density

1. $x(t)$ real$\rightarrow R_x(\tau) = R_x(-\tau) \rightarrow S_x(\omega) = S_x(-\omega)$ is real: $\mathcal{F}\{\text{real, even function}\}$=real,even function$\rightarrow$cosine xform.
2. $R_x(\tau)$ is *positive semidefinite* $\Leftrightarrow S_x(\omega) \geq 0$:
   $\sigma^2_{y(t)=\int f(t)x(t)dt} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)R_x(t-s)f^*(s)dt\,ds \geq 0.$

Proof: $\Rightarrow$: Suppose $\exists\,\omega_o$ so that $S_x(\omega_o) < 0$. Let $f(t) = e^{j\omega_o t}$.
Then: $\sigma_y^2 = \int\int e^{j\omega_o t}R_x(t-s)e^{-j\omega_o s}dt\,ds = S_x(\omega_o)\cdot\infty < 0.$

Proof: $\Leftarrow$: For any real $f(t)$, write $f(t) = \int F(\omega)e^{j\omega t}d\omega$.
Then: $\sigma_y^2 = \int |F(\omega)|^2 S_x(\omega)d\omega \geq 0$ using Parseval twice.
Note: Much simpler in the frequency domain! (just nonnegative)

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3. See table of properties on p.471 of Stark and Woods.
4. *Average power* $= E[x(t)^2] = \sigma^2_{x(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)d\omega$.

Note: "Power" is the *tendency* of random $|x(t)|$ to be large:
Larger variance$\rightarrow$broader pdf$\rightarrow$RV tends to be larger.
EX: Let $v(t)$=random voltage across a resistor $R = 1\Omega$.
Average power=$E[v(t)^2]$; large despite $E[v(t)] = 0$.

EECS 501 PSD OF LINEAR TIME-INVARIANT Fall 2000
(LTI) SYSTEM OUTPUT WITH WSS PROCESS INPUT

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1. LTI system; impulse response $h(t)$:$\delta(t) \rightarrow \boxed{h(t)} \rightarrow h(t)$
2. LTI system has transfer function $H(\omega) = \mathcal{F}\{h(t)\}$:
   $\cos(\omega t) \rightarrow \boxed{h(t)} \rightarrow |H(\omega)| \cos(\omega t + ARG[H(\omega)])$
3. WSS random processes: $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$
4. IMPORTANT FORMULA: $S_y(\omega) = |H(\omega)|^2 S_x(\omega)$.

From: Take $\mathcal{F}$ of $R_y(\tau) = \int\int h(u)h(v)R_x(\tau - u + v)du\,dv$.

Note: Compare to random vectors: $y = Ax \rightarrow K_y = AK_xA^T$.

EX: $x(t) \rightarrow \boxed{\frac{dy}{dt} + ay(t) = x(t)} \rightarrow y(t), \quad a > 0$ so stable.
$x(t)$ is a 0-mean uncorrelated WSS RP. What is $S_y(\omega)$?

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1. 0-mean WSS uncorrelated$\rightarrow R_x(\tau) = \delta(\tau) \rightarrow S_x(\omega) = 1$.
   (Assume WLOG that the area under impulse is unity.)

2a. **Impulse response:** $h(t) = e^{-at}$ for $t \geq 0$; 0 otherwise.

2b. **Transfer function:** $H(\omega) = \mathcal{F}\{h(t)\} = 1/(j\omega + a)$.

3. $S_y(\omega) = |1/(j\omega + a)|^2 \cdot 1 = 1/(\omega^2 + a^2)$.
4. $R_y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2+a^2} e^{j\omega\tau} d\omega = \frac{1}{2a}e^{-a|\tau|}$.
5. $\sigma^2_{y(t)} = E[y(t)^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2+a^2} d\omega = R_y(0) = \frac{1}{2a}$
6. $x(t)$ Gaussian$\rightarrow y(t)$ Gaussian$\rightarrow f_{y(t)}(Y) \sim N(0, \frac{1}{2a})$.
7. See p. 487 of Stark and Woods for more details.

EX: $x(t)$ Gaussian; $a = 5$. $Pr[1 < x(5) < 2\,\&\,3 < x(6) < 4]$=?

Soln: $Pr = \int_1^2 \int_3^4 \frac{1}{2\pi} \frac{1}{\sqrt{det[K]}} e^{-\frac{1}{2}[X_5,X_6]K^{-1}[X_5,X_6]'} dX_6\, dX_5$

where: $K = \frac{1}{10}\begin{bmatrix} 1 & e^{-5} \\ e^{-5} & 1 \end{bmatrix} \rightarrow det[K] = 0.01(1 - e^{-10})$.

Def: A 0-mean WSS RP $x(t)$ is a *white* process if $S_x(\omega) = \sigma^2$ for some positive constant $\sigma^2 > 0$.

Comments on White Processes:

1. All frequencies ω equally represented→"white" process: Red+orange+yellow+green+blue+violet+others=white if all colors (even not listed) present in equal strengths.
2. $R_x(\tau) = \sigma^2\delta(\tau)$; $x(t)$ is an *uncorrelated* process.
3. Power=$R_x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sigma^2 d\omega \to \infty$! (impulse at $\tau = 0$).
   a. *Infinite* power→ white process cannot exist physically!
   b. NOT a 2nd-order process. Still: often used in models.
4. Implications of *continuous*-time uncorrelated RP:
   a. Knowledge of $x(7)$ does not help you predict $x(7.000001)$;
   b. Typical sample function (realization) of a white RP: ∼scatter plot (dust sprinkled on figure with t axis).
   c. Takes *infinite power* to be able to move from $x(7)$ to different value $x(7.000001)$ in *almost-zero* time.

So What Good are White RPs If They Don't Exist?

1a. In modelling: Usually pass white $x(t)$ through LTI system.
1b. Real-life systems have finite bandwidth: $\lim_{\omega\to\infty}|H(\omega)| = 0$.
1c. So $S_x(\omega)$ *doesn't matter* for large ω: gets filtered anyway. White input and bandlimited white input→same output.
2. A 2nd-order 0-mean WSS RP $x(t)$ can be modelled as:

white RP→ $\boxed{|H(\omega) = \sqrt{S_x(\omega)}|}$ → $x(t)$. OR: $H(\omega)$ causal.

$$H(s) = \frac{\prod(s+z_i)(s+z_i^*)}{\prod(s+p_i)(s+p_i^*)} \to S_x(\omega) = \frac{\prod(\omega^2+z_i^2)(\omega^2+z_i^{*2})}{\prod(\omega^2+p_i^2)(\omega^2+p_i^{*2})} = \frac{\prod|\omega^2+z_i^2|^2}{\prod|\omega^2+p_i^2|^2}.$$

using: $(j\omega+z)(-j\omega+z)(j\omega+z^*)(-j\omega+z^*) = (\omega^2+z^2)(\omega^2+z^{*2})$.

1. $x(t) \rightarrow \overline{\underline{|H(\omega)|}} \rightarrow y(t), \quad H(\omega) = \begin{cases} 1, & \text{if } 3 \le |\omega| \le 3.001; \\ 0, & \text{otherwise.} \end{cases}$
2. Then $S_y(\omega) = \begin{cases} S_x(\omega) \approx S_x(3), & \text{if } 3 \le |\omega| \le 3.001; \\ 0, & \text{otherwise.} \end{cases}$
3. Then the average power $E[y(t)^2]$ in output $y(t)$ is:

$$E[y(t)^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(\omega)d\omega = \frac{2}{2\pi}\int_{3}^{3.001} S_x(\omega)d\omega \approx \frac{0.001}{\pi}S_x(3).$$

4. **Interpretation:** $S_x(3)$ *is the average power per unit bilateral bandwidth in* $x(t)$ *at* $\omega = 3$:
   a. **Bilateral:** components at *both* $\omega = 3$ and $\omega = -3$;
   b. $\frac{2\Delta}{2\pi}S_x(\omega_o)$ is the average power in random process $x(t)$ in the frequency band of width Δ: $\omega_o \le \omega \le \omega_o + \Delta$.
   c. **Units:** $x(t)$ volts$\rightarrow \frac{1}{2\pi}S_x(\omega)\ \frac{volts^2}{rad/sec}$; $S_x(f)\ \frac{volts^2}{Hertz}$.
5. $S_x(\omega)$ must be multiplied by frequency to get power in a frequency band; it is a power spectral **density**:
   a. • $S_x(\omega_o)\frac{2\Delta}{2\pi} = Power\,in\,[\omega_o \le \omega \le \omega_o + \Delta]$.
   b. • $S_x(f_o)2\Delta = Power\,in\ [f_o \le f \le f_o + \Delta]$.
   c. **Compare:** $f_x(X)\Delta = Pr[X \le x \le X + \Delta]$.
6. This interpretation makes the following evident:
   a. $S_x(\omega) \ge 0$: otherwise power in some frequency band would be negative! **Compare to:** $f_x(X) \ge 0$.
   b. Total average power=$E[x(t)^2] = \int_{-\infty}^{\infty} S_x(f)df$.
   Compare to: Total probability=$\int_{-\infty}^{\infty} f_x(X)dX = 1$.

1. Recall we can *decorrelate* a random N-vector x as follows:
   a. Covariance matrix K_x has N eigenvalues λ_i and eigenvectors $\phi_i, i = 1 \ldots N$, where $K_x\phi_i = \lambda_i\phi_i$.
   b. Let $A = [\phi_1 \ldots \phi_N]^T$ (don't forget transpose!) and $y = Ax$. Then: $y_i = \phi_i^T x = \phi_i \cdot x = \sum_{j=1}^N (\phi_i)_j x_j$.
   c. Then $K_y = AK_xA^T = DIAG[\lambda_1 \ldots \lambda_N]$ and then $E[y_iy_j] = \lambda_i\delta(i-j) \rightarrow \{y_i\}$ have been decorrelated.
   d. $y = Ax \rightarrow x = A^Ty \rightarrow x = \sum_{i=1}^N y_i\phi_i$: x=sum of uncorrelated RVs×eigenvectors.

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2. Now try this for 2nd-order 0-mean WSS processes:
   a. Eigen*functions* of LTI systems: $\phi(t) = e^{j\omega t}$: Means
   b. $e^{j\omega t} \rightarrow \boxed{|H(\omega)|} \rightarrow H(\omega)e^{j\omega t} = |H(\omega)|e^{(j\omega t + ARG[H(\omega)])}$.
3. $x(t)$ is a real-valued 2nd-order 0-mean WSS process.
   a. Define RVs $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}$, for all ω.
   b. Then $\{X(\omega)\}$ are *uncorrelated random variables*:

   $$E[X(\omega_1)X^*(\omega_2)] = 2\pi S_x(\omega_1)\delta(\omega_1 - \omega_2).$$
4. Spectral interpretation of 2nd-order WSS RPs:

   $x(t) = \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\frac{d\omega}{2\pi}$: $\int$ uncorrelated RVs×eigenfunctions.
5. We have for *finite but large* T and interval $[-\frac{T}{2}, \frac{T}{2}]$:

K-L: $x(t) = \sum_{n=-\infty}^{\infty} x_n \frac{1}{\sqrt{T}} e^{j2\pi nt/T}, \quad |t| \leq T/2 \rightarrow \infty$,

where: $x_n = \int_{-T/2}^{T/2} x(t)\frac{1}{\sqrt{T}}e^{-j2\pi nt/T}dt$ for integers n.

Then: $E[x_ix_j] = S_x(2\pi i/T)\delta(i-j) \rightarrow \{x_n\}$ uncorrelated.

DEF: This is *Karhunen-Loeve expansion* for WSS processes.

Works: $S_x(\omega) \approx 0$ for $|\omega| > B \rightarrow$ need $TB >> 1$.

1. $E[X(\omega_1)X^*(\omega_2)] = E[\int x(t)e^{-j\omega_1 t}dt \int x(s)e^{j\omega_2 s}ds] =$
   $\int\int E[x(t)x(s)]e^{-j(\omega_1 t-\omega_2 s)}dt\,ds.$ $E[x(t)x(s)] = R_x(t-s).$
2. *Change variables:* $t, s \to \tau = t-s, z = t+s : |J| = 2.$
   $E[X(\omega_1)X^*(\omega_2)] = \int\int R_x(\tau)e^{-j[\omega_1(\tau+z)+\omega_2(\tau-z)]/2}\frac{d\tau\,dz}{2}$
   $= \int R_x(\tau)e^{-j\left(\frac{\omega_1+\omega_2}{2}\right)\tau}d\tau \int e^{-j\left(\frac{\omega_1-\omega_2}{2}\right)z}\left(\frac{dz}{2}\right)[\mathcal{F}\{1\} = 2\pi\delta(\omega)]$
   $= 2\pi S_x\left(\frac{\omega_1+\omega_2}{2}\right)\delta(\omega_1-\omega_2) = 2\pi S_x(\omega_1)\delta(\omega_1-\omega_2).$ QED.

Gaussian RPs and The Distribution of $X(\omega)$

1. Let $x(t)$ be Gaussian RP$\to$ $X(\omega)$ Gaussian RVs:
   a. $Re[X(\omega)]$ and $Im[X(\omega)]$ are Gaussian RVs;
   b. $|X(\omega)|$ Rayleigh RVs; $ARG[X(\omega)]$ uniform RVs.
   Rayleigh pdf: $f_z(Z) = \frac{Z}{\sigma^2}e^{-Z^2/(2\sigma^2)}, Z \geq 0.$ p. 138.

2a. **Physical** Let $H(\omega) = \delta(\omega-\omega_o) + \delta(\omega+\omega_o)$:
   interpretation: $H(\omega)$ passes ONLY frequency ω_o.

   b. $x(t) \to \boxed{|H(\omega)|} \to y(t);$ $H(\omega)$ narrowband.

3. All possible sample functions of RP $x(t)$ are filtered.
   a. $y(t) = A\cos(\omega_o t) + B\sin(\omega_o t)$ for RVs A, B.
   b. Jointly Gaussian RVs $A, B \sim N(0, S_x(\omega_o)\pi\infty)$.

REF: Papoulis, 3^{rd} ed. (1991), pp. 416-418.

Let: $x(t)$ and $y(t)$ be real-valued 0-mean *jointly* WSS RPs.
with: Cross-correlation $R_{xy}(\tau) = E[x(t)y(t-\tau)]$ for any t.
Means: $E[x(t)y(s)] = R_{xy}(t-s)$; jointly WSS→not t, s separately.

Def: The *Cross-Spectral Density* $S_{xy}(\omega)$ is defined as:
$S_{xy}(\omega) = \mathcal{F}\{R_{xy}(\tau)\}; \quad R_{xy}(\tau) = \mathcal{F}^{-1}\{S_{xy}(\omega)\}$.

Properties of The Cross-Spectral Density

1. $R_{yx}(\tau) = R_{xy}(-\tau) \rightarrow S_{yx}(\omega) = S_{xy}(-\omega) = S^*_{xy}(\omega)$.
2. $S_{xy}(\omega)$, unlike $S_x(\omega)$, is *not* a real or even function.
3. WSS random processes: $x(t) \rightarrow \boxed{H(\omega)} \rightarrow y(t)$

   FORMULA: $S_{yx}(\omega) = H(\omega)S_x(\omega)$ (note S_{yx}, not S_{xy}):
   Take $\mathcal{F}$ of $R_{yx}(\tau) = \int h(u)R_x(\tau - u)du$.
4. From #1 and #3 we have $S_{xy}(\omega) = H^*(\omega)S_x(\omega)$.
5. Exchange $x(t), y(t)$ and replace $H(\omega)$ with $\frac{1}{H(\omega)}$:
   $S_{yx}(\omega) = \frac{1}{H^*(\omega)}S_y(\omega) \rightarrow S_y(\omega) = |H(\omega)|^2 S_x(\omega)$!

Example of Cross-Spectral Density

Let $x(t) \rightarrow \boxed{F(\omega)} \rightarrow y(t)$ and $x(t) \rightarrow \boxed{G(\omega)} \rightarrow z(t)$
Compute the cross-spectral density $S_{yz}(\omega)$. **Solution:**

1. $x(t), y(t), z(t)$ are real-valued 0-mean jointly WSS RPs.
2. $y(t) = \int f(u)x(t-u)du$ and $z(s) = \int g(v)x(s-v)dv \rightarrow$
3. $E[y(t)z(s)] = \int\int f(u)g(v)E[x(t-u)x(s-v)]du\,dv \rightarrow$
   $R_{yz}(t-s) = \int\int f(u)g(v)R_x(t-s-u+v)du\,dv \rightarrow$
   $S_{yz}(\omega) = F(\omega)G^*(\omega)S_x(\omega)$. Neat formula! (I think so)
4. Passbands of $F(\omega)$ and $G(\omega)$ don't overlap→ $R_{yz}(\tau) = 0$.
   $y(t)$ and $z(t)$ (different frequency components of $x(t)$)
   are *uncorrelated*→spectral interpretation of WSS RPs.

Infinite Smoothing Filter for Signal in Noise:

1. $y(t), x(t), v(t)$ are 2nd-order 0-mean jointly WSS RPs.
   a. Observe $y(t) = x(t) + v(t)$ where $E[x(t)v(s)] = 0$.
   b. $x(t)$=signal, $v(t)$=noise, $y(t)$=noisy data.
2. We have the following (cross)covariance functions:
   a. $R_{xy} = E[x(t)y(s)] = E[x(t)(x(s) + v(s))] = R_x$.
   b. $R_y = E[(x(t) + v(t))(x(s) + v(s))] = R_x + R_v$.
3. GOAL: Compute LLSE of $x(t)$ from $\{y(s), -\infty < s < \infty\}$.
   NOTE: $x(t), v(t)$ jointly Gaussian RPs$\rightarrow \hat{x}_{LLSE} = \hat{x}_{LS}$.

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4. LEMMA: 0-mean *uncorrelated* $\{x(i)\}$ and $\{v(i)\}$.
   Observe $y(n) = x(n) + v(n)$ where $E[x(i)v(j)] = 0$.
   Then LLSE of $x(i)$ from $\{y(j), -\infty < j < \infty\}$ is
   $\hat{x} = K_{xy}K_y^{-1}y = K_x(K_x + K_v)^{-1}y$
   $= DIAG[\sigma^2_{x(i)}]DIAG[\sigma^2_{x(i)} + \sigma^2_{v(i)}]^{-1}y$
   $\rightarrow \hat{x}(i) = \frac{\sigma^2_{x(i)}}{\sigma^2_{x(i)}+\sigma^2_{v(i)}}y(i)$ where y=vector of $\{y(j)\}$.
   Problem *decouples* since the $\{y(j)\}$ are uncorrelated.

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5. Write $x(t) = \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\frac{d\omega}{2\pi}$; write $y(t), v(t)$ similarly.
   Apply LEMMA to RVs $X(\omega), Y(\omega), V(\omega)$. **Solution:**
   $y(t) \rightarrow \boxed{\frac{S_x(\omega)}{S_x(\omega)+S_v(\omega)}} \rightarrow \hat{x}(t)$.
6. Comments:
   a. More insightful than Recitation derivation!
   b. Shows significance of decorrelation=prewhitening:
   Eliminates need to compute K_y^{-1} (saves much work).
   c. See Stark and Woods p. 553 for more details.