DEF: *Bernoulli* random process $x(n)$ is a discrete-time 1-sided iidrp with:

$$x(n) = \begin{cases} 1 & \text{success or arrival with prob. } p \\ 0 & \text{failure or nonarrival with } 1-p \end{cases} \quad p_{x(n)}(X) = \begin{cases} p & \text{for } X=1 \\ 1-p & \text{for } X=0 \end{cases}$$

Note: Kolmogorov: $p_{x(i_1)\dots x(i_N)}(X_1 \dots X_N) = \prod_{i=1}^{N} p_{x(n)}(X_i)$. Bernoulli rvs.

Question	pmf name	pmf formula	$\mathbf{E[k]}$	$\sigma^2_{\mathbf{k}}$
$Pr[\substack{\text{k successes}\\ \text{in N trials}}]$	Binomial	$\binom{N}{k}p^k(1-p)^{N-k}$	Np	$Np(1-p)$
$[\substack{\text{\#trials until}\\ \text{next success}}]$	Geometric	$(1-p)^{k-1}p, k \geq 1$	$1/p$	$(1-p)/p^2$
$[\substack{\text{\#trials until}\\ \text{r}^{\text{th}} \text{ success}}]$	Pascal	$\binom{k-1}{r-1}p^r(1-p)^{k-r}$	r/p	$r(1-p)/p^2$

Note: "Until" means "up to *and including*" in the above. pmf ranges omitted.
Binomial: Pr[k successes in any *closed* interval of length $N-1$ (N points)]
Binomial: =sum of N independent Bernoulli rvs.: z-xform=$((1-p)+pz)^N$.

Geometric: 1^{st}-order interarrival time=#trials from last success to next success.
Geometric: Let A=next success on k^{th} trial and B_j=no successes on last j trials.
Memoryless: $Pr[A|B_j] = \frac{Pr[AB_j]}{Pr[B_j]} = \frac{Pr[B_{k+j-1}]p}{Pr[B_j]} = \frac{(1-p)^{k+j-1}p}{(1-p)^j} = \underset{k=1,2\dots}{(1-p)^{k-1}p} = Pr[A]$.

Pascal: r^{th}-order interarrival time=sum of r independent Geometric rvs.
Pascal: Pr[$r-1$ successes in $k-1$ trials]Pr[r^{th} success in k^{th} trial]. $k \geq r$.

DEF: *Poisson* process: continuous-time with arrivals at points in time.

1. $Pr[\text{arrival in } [t_o, t_o+\delta t]] = \lambda\delta t$ as $\delta t \to 0$. λ=*average* arrival rate.
2. Events defined on non-overlapping intervals are independent.
3. Continuous-time limit of Bernoulli with $p = \lambda\delta t$ and $N = T/\delta t$.

Question	pdf name	pdf formula	$\mathbf{E[t]}$	$\sigma^2_{\mathbf{t}}$
$Pr[\substack{\text{k arrivals}\\ \text{in time T}}]$	Poisson **pmf**	$(\lambda T)^k e^{-\lambda T}/k!$	λT	λT
$[\substack{\text{time t until}\\ \text{next arrival}}]$	Exponential	$\lambda e^{-\lambda t}, t \geq 0$	$1/\lambda$	$1/\lambda^2$
$[\substack{\text{time t until}\\ \text{r}^{\text{th}} \text{ arrival}}]$	Erlang	$\lambda^r t^{r-1}e^{-\lambda t}/(r-1)!$	r/λ	r/λ^2

Poisson: $\binom{N}{k}p^k(1-p)^{N-k} \approx \frac{N^k}{k!}p^k(1-p)^N \to (T/\delta t)^k(\lambda\delta t)^k(1-\lambda\delta t)^{T/\delta t}/k!$.
Exponen: $(1-p)^{k-1}p \to (1-\lambda\delta t)^{t/\delta t}(\lambda\delta t) \to \lambda e^{-\lambda t}\delta t$ since $\lim_{x\to 0}(1+ax)^{b/x} = e^{ab}$.
Exponen: *Memoryless*, like Geometric pmf (similar derivation to that above).
Erlang: r^{th}-order interarrival time=sum of r independent Exponential rvs.
Counting: Poisson *counting* process $N(t)$=#arrivals in Poisson process in $[0,t]$.
Refs: pp. 377-384 and 36-42; also see A.W. Drake text on closed reserve.

$\sum$: x_1, x_2 *independent* Poisson processes with avg. arrival rates λ_1, λ_2.
DEF: New rp x_3 where an arrival in *either* x_1 *or* x_2 is an arrival in x_3.
Then: x_3 is *also* a Poisson process with avg. arrival rate $\lambda_3 = \lambda_1 + \lambda_2$,
since: $Pr[\text{arrival in } [t_o, t_o + \delta t]] = \lambda_1\delta t + \lambda_2\delta t - \lambda_1\lambda_2(\delta t)^2 \to (\lambda_1 + \lambda_2)\delta t$,
and events defined on non-overlapping intervals are still independent.

EX: $x_1 \dots x_N$ are iidrvs with exponential pdf $f_{x_i}(X) = \lambda e^{-\lambda X}, X \geq 0$.
Then: $y = \min[x_1 \dots x_N]$ has exponential pdf $f_y(Y) = N\lambda e^{-N\lambda Y}, Y \geq 0$
since: y is 1^{st} arrival in superposition of N indpt Poisson processes.

EX: 8 light bulbs turned on at $t = 0$. Bulb lifetime is an exponential pdf.
Q: Compute mean and variance of time t until the 3^{rd} bulb burns out.
A: Bulb burnout=arrival in Poisson process (only until it burns out!).
$\sum$: Sum of n independent Poisson processes (n=#bulbs still on).

$E[t]$: $E[t] = 1/(8\lambda) + 1/(7\lambda) + 1/(6\lambda)$. $\quad \sigma_t^2 = 1/(8\lambda)^2 + 1/(7\lambda)^2 + 1/(6\lambda)^2$.

Q: In x_3, compute Pr[next arrival comes from x_1, as opposed to x_2].

A1: $Pr[\text{arrival } x_1 | \text{arrival } x_3] = \frac{Pr[\text{arrival } x_1 \,\&\, x_3]}{Pr[\text{arrival } x_3]} = \frac{\lambda_1\delta t}{(\lambda_1+\lambda_2)\delta t} = \frac{\lambda_1}{\lambda_1+\lambda_2}$.

A2: t_i=time to next arrival in x_i. $f_{t_i}(T_i) = \lambda_i e^{-\lambda_i T_i}, T_i \geq 0, i = 1, 2$.
Want: $Pr[t_1 < t_2] = \int_0^\infty \int_{T_1}^\infty \lambda_1 e^{-\lambda T_1} \lambda_2 e^{-\lambda T_2} dT_2 \, dT_1 = \frac{\lambda_1}{\lambda_1+\lambda_2}$.

Note: Pr[7 of next 10 arrivals in x_3 from x_1]=$\binom{10}{7}(\frac{\lambda_1}{\lambda_1+\lambda_2})^7(\frac{\lambda_2}{\lambda_1+\lambda_2})^3$.

Random erasures x is a Poisson process with average arrival rate λ.
At each arrival in x, flip a coin with Pr[heads]=P.
If heads: Count the arrival in x as an arrival in a new process y.
If tails: *Don't* count arrival in x as an arrival in new process y.
Assume: Coin flips are independent, and flipping is independent of x.
Then: y is a Poisson process with average arrival rate λP.
EX: Defective Geiger counter only works with Pr[detect particle]=P.
Radioactivity is well-modelled by Poisson process: arrivals=particles.

But: If coin flips *not* independent, y is not Poisson.
EX: If coin alternates heads and tails, not *random* erasures.
Then: Interarrival times for y are 2^{nd}-order Erlang pdf!