Given: $f_{x,y}(X,Y) = 6e^{-(3X+2Y)}$ for $X,Y \geq 0$; 0 otherwise (2-D exponential).
Goal: Compute $f_{z,w}(Z,W)$ for transformation $z = x+y$ and $w = x/(x+y)$.

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1. Compute the *inverse transformation* of the given one:

$$\left\{ \begin{array}{c} z = z(x,y) = x+y \\ w = w(x,y) = x/(x+y) \end{array} \right\} \rightarrow \text{Inverse} \left\{ \begin{array}{c} x = x(z,w) = zw \\ y = y(z,w) = z(1-w) \end{array} \right\}$$

since $w = x/(x+y) = x/z \rightarrow x = zw$ and $y = z-x = z-zw = z(1-w)$.

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2. Compute the *Jacobian*=determinant of the *Jacobian matrix* J:

$$|J| = |det \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}| = |det \begin{bmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{bmatrix}| = |-\frac{1}{x+y}| = \frac{1}{x+y}.$$

since $x,y \geq 0$ for this particular $f_{x,y}(X,Y)$.

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3. Substitute inverse transformation into $f_{x,y}(X,Y)/|J(X,Y)|$:

$f_{z,w}(Z,W) = \frac{f_{x,y}(X,Y)}{|J(X,Y)|}|_{X=x(Z,W),Y=y(Z,W)}$ as defined above

$= \frac{f_{x,y}(ZW,Z(1-W))}{1/(ZW+Z(1-W))} = 6Ze^{-(3ZW+2Z(1-W))} = 6Ze^{-Z(W+2)}$

for $Z \geq 0$ and $0 \leq W \leq 1$ since $x,y \geq 0 \rightarrow 0 \leq w \leq 1$.

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4. If desired, compute marginal pdfs for z and/or w:

$f_z(Z) = 6Ze^{-2Z}\int_0^1 e^{-ZW}dW = 6(e^{-2Z} - e^{-3Z})$ for $Z \geq 0$.

$f_w(W) = \int_0^\infty 6Ze^{-Z(W+2)}dZ = \frac{6}{(W+2)^2}$ for $0 \leq W \leq 1$.

Check: Both marginal pdfs integrate to 1. Compare to *exponential* $f_x(X), f_y(Y)$.

Given: $f_{x,y}(X,Y) = 9e^{-(3X+3Y)}$ for $X,Y \geq 0$; 0 otherwise (2-D exponential).
Goal: Compute $f_{z,w}(Z,W)$ for transformation $z = x+y$ and $w = x/(x+y)$.

Now get $f_{z,w}(Z,W) = 9Ze^{-3Z}$ for $Z \geq 0$ and $0 \leq W \leq 1$.

This is a 2^{nd}-*order Erlang* or *Gamma* pdf in z; a *uniform* pdf in w.

Note that z and w are now *independent* random variables, unlike before.