EECS 501 RESULTS SUMMARY: ERGODICITY Fall 2001

ISSUE: Let $\{x_i, i = 1, 2\ldots\}$ be a sequence of id rvs. Does the *sample mean* $M_n = \frac{1}{n}\sum_{i=1}^{n} x_i$ converge to the *ensemble mean* $E[x_i] = \mu$, and in what sense? "id"="identically distributed"; assume $E[x_i], \sigma^2_{x_i} < \infty$.

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1. **Weak Law of Large Numbers:**
   $\{x_i\}$ are *independent*$\rightarrow$ ($M_n \rightarrow \mu$ *in probability*).
   PROOF: Lecture in Oct.; "Convergence of RVs" handout.
   Equivalent to: "M_n is a *weakly consistent* estimator of μ."

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2. **Mean Ergodic Theorem:**
   $\{x_i\}$ are *independent*$\rightarrow$ ($M_n \rightarrow \mu$ *in mean square*):
   PROOF: "Convergence of RVs" handout. $\underset{n\rightarrow\infty}{L.I.M.} M_n = \mu$.

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3. **Strong Law of Large Numbers:**
   $\{x_i\}$ are *independent*$\rightarrow$ ($M_n \rightarrow \mu$ *with probability one*).
   PROOF: See "Strong Law of Large Numbers" handout.

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4. $\{x_i\}$ have **finite correlation length**:
   $K_x(i,j) = 0$ if $|i - j| > M$ for some $M < \infty$
   $\rightarrow$ ($M_n \rightarrow \mu$ *in probability*). PROOF: Exam #2, Fall 1998.

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5. $\{x_i\}$ has $\underset{n\rightarrow\infty}{LIM} \frac{1}{n}\sum_{i=1}^{n} K_x(i,n) = 0$.
   $\rightarrow$ ($M_n \rightarrow \mu$ *in mean square*): $\underset{n\rightarrow\infty}{L.I.M.} M_n = \mu$.
   PROOF: Problem Set #8 (adapted to a nonzero mean μ).

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6. $\{x_i\}$ **asymptotically uncorrelated:** $\underset{|n|\rightarrow\infty}{LIM} K_x(n) = 0$.
   $\rightarrow$ ($M_n \rightarrow \mu$ **in mean square**): $\underset{n\rightarrow\infty}{L.I.M.} M_n = \mu$.
   PROOF: Stark and Woods p. 449 (cont.-time version).
   Makes sense: group into sum of uncorrelated sums of RVs.

GIVEN: Observations of continuous-time RP $\{x(t), t > 0\}$, where $x(t)$ fulfills any of the ergodicity conditions overleaf.

1. Use $\frac{1}{t}\int_0^t x(s)ds$ to estimate $\mu = E[x(t)]$.
   Polling (see Problem Set #5). Now assume WLOG $\mu = 0$.
2. Use $\frac{1}{t}\int_0^t x^2(s)ds$ to estimate $\sigma^2_{x(t)} = E[x^2(t)]$. Need $E[x^2(t)], E[x^4(t)] < \infty$ (use Gaussian moment factoring).
3. **Spectral estimation** using the **Periodogram:** Note
$X(\omega) = \int x(t)e^{-j\omega t}dt \rightarrow E[X(\omega_1)X^*(\omega_2)] = S_x(\omega_1)\delta(\omega_1-\omega_2)$.
   Suggests estimating $S_x(\omega)$ by estimating $E[|X(\omega)|^2]$ using:

DEF: Periodogram$= P = \frac{1}{T}|\int_0^T x(t)e^{-j\omega t}dt|^2$ (note units).

THM: P is *asymptotically unbiased* estimator of $S_x(\omega)$.

1. $E[P] = E[\frac{1}{T}\int_0^T x(t)e^{-j\omega t}dt \int_0^T x(s)e^{j\omega s}ds]$. Simplify:
2. $E[P] = \frac{1}{T}\int_0^T\int_0^T R_x(t-s)e^{-j\omega(t-s)}dt\,ds$ (looks familiar).
3. Change variables: $t, s \rightarrow \tau = t-s, z = t+s : |J| = 2$.
4. $E[P] = \frac{1}{T}\int_{-T}^T d\tau R_x(\tau)e^{-j\omega\tau}\int_{|\tau|}^{2T-|\tau|}\frac{dz}{2}$
   $E[P] = \int_{-T}^T R_x(\tau)e^{-j\omega\tau}(1-\frac{|\tau|}{T})d\tau$.
5. Now take $\underset{T\rightarrow\infty}{LIM} E[P] = \underset{T\rightarrow\infty}{LIM}\int_{-T}^T R_x(\tau)e^{-j\omega\tau}(1-\frac{|\tau|}{T})d\tau$
   $= \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau}d\tau = S_x(\omega)$ provided $\underset{|\tau|\rightarrow\infty}{LIM} R_x(\tau) = 0$.
6. Periodogram is **not** *consistent* estimator: $\sigma_P^2 \approx S_x^2(\omega)$.
   $\sigma_P \approx E[P]$ *regardless* of data length! So $\underset{T\rightarrow\infty}{LIM} P \neq S_x(\omega)$.
7. See Stark and Woods p. 472 and p. 494.

EXAMPLE OF NON-ERGODICITY

1. Begin by flipping coin A with $Pr[heads] = P = 0.5$.

2a. Heads$\rightarrow$use coin B with $P = 0.7$ $\rightarrow$Bernoulli process.

2b. Tails$\rightarrow$use coin C with $P = 0.8$ $\rightarrow$Bernoulli process.

3. $M_n \rightarrow$0.7 or 0.8, but $\mu = 0.75$ for this random process!

THM: Let $\{x_i, i = 1, 2 \ldots\}$ be a sequence of iidrvs with finite mean $\mu = E[x_i]$ and finite variance $\sigma^2_{x_i}$. $M_n = \frac{1}{n}\sum_{i=1}^{n} x_i$ converges *with probability one* to μ.

PROOF: Thm. 3 from "Convergence of RVs" handout:

1. If $\sum_{n=1}^{\infty} Pr[|M_n - \mu| > \epsilon] < \infty$ then $M_n \to \mu$ *with prob 1.* This followed from the *Borel-Cantelli Lemma.*

Q. Can we use Chebyschev inequality directly, as before?

A. $Pr[|M_n - \mu| > \epsilon] < \frac{\sigma^2_{M_n}}{\epsilon^2} = \frac{1}{n}\frac{\sigma^2_x}{\epsilon^2}$ but $\frac{\sigma^2_x}{\epsilon^2}\sum_{n=1}^{\infty}\frac{1}{n} \to \infty$.

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2. Change variables from n to $m = [\sqrt{n}]$ (e.g., $[3.6] = 3$) and x_i to $\tilde{x}_i = x_i - \mu$, so that $E[\tilde{x}_i] = 0$:

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3. $\sum_{i=1}^{n}\tilde{x}_i = \sum_{i=1}^{m^2}\tilde{x}_i + \sum_{i=m^2+1}^{n<(m+1)^2}\tilde{x}_i$.
   EX: $x_1 + \ldots x_{27} = (x_1 + \ldots x_{25}) + (x_{26} + x_{27})$; $\quad 5 = [\sqrt{27}]$

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4. $m^2 \le n \to |M_n| \le |M_{m^2}| + \frac{1}{m^2}|\sum_{i=m^2+1}^{n}\tilde{x}_i|$.

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5. $Pr[|M_{m^2}| > \epsilon] < \frac{\sigma^2_{M_{m^2}}}{\epsilon^2} = \frac{1}{m^2}\frac{\sigma^2_x}{\epsilon^2}$ as above,
   but now $\frac{\sigma^2_x}{\epsilon^2}\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\sigma^2_x}{\epsilon^2}\frac{\pi^2}{6} < \infty$!
   So $M_{m^2} \to 0$ with probability one.

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6. $Pr[\frac{1}{m^2}|\sum_{i=m^2+1}^{n}\tilde{x}_i| > \epsilon] < \frac{(n-m^2)\sigma^2_x}{m^4\epsilon^2} < \frac{(2m+1)\sigma^2_x}{m^4\epsilon^2}$
   since $m^2 = [\sqrt{n}]^2 \le n < ([\sqrt{n}]+1)^2 = m^2 + (2m+1)$.

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7. But $\sum_{m=1}^{\infty}\frac{(2m+1)\sigma^2_x}{m^4\epsilon^2} < \sum_{m=1}^{\infty}\frac{\sigma^2_x}{m^2\epsilon^2} = \frac{\pi^2}{6}\frac{\sigma^2_x}{\epsilon^2} < \infty$.
   So $\frac{1}{m^2}\sum_{i=m^2+1}^{n}\tilde{x}_i \to 0$ with probability one.

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8. From the bound in #4, $M_n \to 0$ with probability one.

EECS 501 Fall 2001

POISSON RANDOM SAMPLING OF SIGNALS

THM: Let $x(t)$ be *deterministic* with $E = \int x^2(t)dt < \infty$. Sample $x(t)$ at the arrival times t_n of a Poisson process. Then $\hat{X}(\omega) = \frac{1}{\lambda}\sum x(t_n)e^{-j\omega t_n}$ is an *unbiased, weakly consistent* (as $\lambda \to \infty$) estimator of $X(\omega) = \int x(t)e^{-j\omega t}dt$.

1a. $p(t)$ =Poisson counting process with avg. arrival rate λ.

1b. $z(t) = \sum \delta(t - t_n)$ where t_n are Poisson arrival times.

2. Then $p(t) \to |\text{Differentiator} \quad d/dt| \to z(t)$
3. Also note $\hat{X}(\omega) = \int x(t)e^{-j\omega t}\frac{z(t)}{\lambda}dt$.

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4. $E[z(t)] = E\left[\frac{dp}{dt}\right] = \frac{d}{dt}E[p(t)] = \frac{d}{dt}(\lambda t) = \lambda$.
5. $E[\hat{X}(\omega)] = \int x(t)e^{-j\omega t}\frac{E[z(t)]}{\lambda}dt = X(\omega)$.
   Thus $\hat{X}(\omega)$ is an *unbiased* estimator of $X(\omega)$.

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6. Let $\tilde{z}(t) = z(t) - \lambda$ and $\tilde{p}(t) = p(t) - \lambda t$; both 0-mean.
7. $K_z(t,s) = E[\tilde{z}(t)\tilde{z}(s)] = E\left[\frac{\partial \tilde{p}(t)}{\partial t}\frac{\partial \tilde{p}(s)}{\partial s}\right]$
   $= \frac{\partial}{\partial t}\frac{\partial}{\partial s}K_p(t,s) = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\lambda MIN[t,s] = \lambda\delta(t-s)$.

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8. $\sigma^2_{\hat{X}(\omega)} = \frac{1}{\lambda^2}\int\int x(t)x(s)K_z(t,s)dt\,ds$
   $= \frac{1}{\lambda^2}\int\int x(t)x(s)\lambda\delta(t-s)dt\,ds = \frac{E}{\lambda}$.
9. $\underset{\lambda\to\infty}{LIM}\,\sigma^2_{\hat{X}(\omega)} = 0 \to \hat{X}(\omega)$ is a *consistent* estimator of $X(\omega)$.

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1. A. Papoulis, *Probability, etc.*, 3rd ed., p. 383-4.
2. F.J. Beutler, "Alias Free Randomly Timed Sampling of Stochastic Processes," *IEEE Trans. Info. Th.*, 1970.
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