DEF: A *discrete-time random process=random sequence* $x(n)$ is mapping $x(n,\omega):(\mathcal{Z}\times\Omega)\to\mathcal{R}$ where Ω=sample space and $\mathcal{Z}=\{integers\}$.

1. Fix $n_o\in\mathcal{Z}\to x(n_o,\omega)$=random variable indexed by n_o.
2. Fix $\omega_o\in\Omega\to x(n,\omega_o)$=sample function=realization.
3. Can think of $x(n)$ as a random vector of infinite length.

THM: *Kolmogorov Extension Thm.:* Discrete-time random process $x(n)$ is completely specified by its joint pdfs $f_{x(i_1)\ldots x(i_N)}(X_1\ldots X_N)$.

EX: An *iidrp* (independent identically distributed random process) has $f_{x(i_1)\ldots x(i_N)}(X_1\ldots X_N)=f_x(X_1)f_x(X_2)\cdots f_x(X_N)$ for any $i_1\ldots i_N$.

DEF: $x(n)$ is N^{th}*-order stationary* if joint pdfs of order N have: $f_{x(i_1)\ldots x(i_N)}(X_1\ldots X_N)=f_{x(i_1+j)\ldots x(i_N+j)}(X_1\ldots X_N)$ for any j.

Means: Shifting time origin does not affect marginal pdfs of order N.

EX: 1^{st}-order stationary $\Leftrightarrow f_{x(i)}(X)=f_{x(j)}(X)\Leftrightarrow x(n)$ idrp (not iidrp).

THM: N^{th}-order stationary$\to$ $(N-K)^{th}$-order stationary for $0\le K\le N-1$.

Proof: Integrate marginals of order N K times$\to$marginals of order $N-K$.

DEF: $x(n)$ SSS *strict sense stationary* $\Leftrightarrow N^{th}$-order stationary for all N.

EX: iidrp $x(n)$ is SSS since $f_{x(i_1)\ldots x(i_N)}(X_1\ldots X_N)=f_x(X_1)\cdots f_x(X_N)$.

DEF: *Mean* $\mu(n)=E[x(n)]$. *Variance* function $\sigma^2_{x(n)}=K_x(n,n)$ where:

DEF: *(Auto)covariance* $K_x(i,j)=E[x(i)x(j)]-E[x(i)]E[x(j)]=\lambda_{x(i),x(j)}$.

DEF: *(Auto)correlation* $R_x(i,j)=E[x(i)x(j)]=K_x(i,j)$ if $x(n)$ is 0-mean.

DEF: *Cross-covariance* $K_{xy}(i,j)=E[x(i)y(j)]-E[x(i)]E[y(j)]=K_{yx}(j,i)$.

DEF: $x(n)$*uncorrelated* $\Leftrightarrow K_x(i,j)=0$ for $i\neq j$. $K_x(i,i)$ may vary with i.

---

1. $K_x(i,i)=\sigma^2_{x(i)}\ge 0$. (2.) $K_x(i,j)=K_x(j,i)$ (symmetry).
3. $|K_x(i,j)|\le\sqrt{K_x(i,i)K_x(j,j)}$ (Schwarz inequality).
4. $\sum_{i=1}^N\sum_{j=1}^N a_iK_x(n_i,n_j)a_j\ge 0$ for any n_i,n_j,N (psd function).

DEF: $x(n)$ WSS *wide sense stationary* $\Leftrightarrow\mu(n)=\mu$ and $K_x(i,j)=K_x(i-j)$.

Props: (1) $K_x(0)=\sigma^2_{x(n)}\ge 0$; (2) $K_x(i)=K_x(-i)$; (3) $|K_x(i)|\le K_x(0)$.

Note: iid$\to$SSS$\to$ N^{th}-order$\to$ 2^{nd}-order$\to$WSS$\to$ 1^{st}-order$\leftrightarrow$id.

DEF: $x(n)$ *Gaussian* $\leftrightarrow\{x(i_1),x(i_2)\ldots x(i_N)\}$ JGRV for all $i_1\ldots i_N$.

Note: For Gaussian rp: (1) Kolmogorov specified; (2) SSS$\Leftrightarrow$WSS.

LTI: A discrete-time system is LTI *linear time-invariant* if its response to input $x(n)$ is output $y(n) = \sum_{i=-\infty}^{\infty} h(i)x(n-i) = \sum_{i=-\infty}^{\infty} h(n-i)x(i)$ where $h(n)$=*impulse response* of system: $x(n) = \delta(n) \to y(n) = h(n)$.

DEF: *Random*$\to$ $y(n,\omega) = \sum h(n-i)x(i,\omega)$ for each $\omega \in \Omega$=sample space.

Then: $E[y(n)] = \sum_{i=-\infty}^{\infty} h(n-i)E[x(i)] = \sum_{i=-\infty}^{\infty} h(i)E[x(n-i)]$.

$K_y(m,n) = \sum\sum h(m-i)h(n-j)K_x(i,j) = \sum\sum h(i)h(j)K_x(m-i,n-j)$.

and: $K_{xy}(m,n) = \sum_{i=-\infty}^{\infty} h(i)K_x(m,n-i) = \sum_{i=-\infty}^{\infty} h(n-i)K_x(m,i)$.

1. System BIBO stable and $\mu(n), K_x(n,n) < \infty$ $\to$these well-defined.
2. $x(n)$ Gaussian$\to$ $y(n)$ Gaussian$\to$only need $E[y(n)]$ and $K_y(m,n)$.

Note: $x(n)$ WSS$\to$ $E[x(i)] = \mu$ and $K_x(i,j) = K_x(i-j)$. Above simplify to:

- $E[y(n)] = \sum_{i=-\infty}^{\infty} h(i)\mu = H(e^{j0})\mu$=constant.
- $K_y(m,n) = \sum\sum h(i)h(j)K_x((m-i)-(n-j))$
  $= \sum\sum h(i)h(j)K_x((m-n)-i+j) = K_y(m-n)$. $y(n)$ is also WSS.
- $K_{xy}(m,n) = \sum h(i)K_x(m-n+i) = K_{xy}(m-n)$. x, y jointly WSS.

Transfer function: $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$. $h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$.

PSD: $S_x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} K_x(n)e^{-j\omega n} = K_x(0) + 2\sum_{n=1}^{\infty} K_x(n)\cos(\omega n)$.

Then: $S_y(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega})S_x(e^{j\omega}) = |H(e^{j\omega})|^2 S_x(e^{j\omega})$. Useful later!

DEF: A *1-sided* discrete-time rp $x(n)$ is defined only for times $n = 0, 1 \ldots$

DEF: A 1-sided rp $x(n)$ is II$\Leftrightarrow$it has *(stationary) independent increments* $\Leftrightarrow$ $\{x(i_1)-x(0), x(i_2)-x(i_1), x(i_3)-x(i_2)\ldots\}$ are independent rvs for all $0 < i_1 < i_2 < \ldots$ and pdf of $x(i_1) - x(i_2)$ depends only on $i_1 - i_2$.

THM: $y(n)$ is II, y(0)=0 $\Leftrightarrow$ $y(n) = \sum_{i=1}^{n} x(i)$ for some iidrp $x(n)$. **Proof:**

$\Rightarrow$: $x(n) = y(n) - y(n-1) \to x(n)$ iidrp and $y(n) = \sum_{i=1}^{n} x(i)$.

$\Leftarrow$: $y(n) = \sum_{i=1}^{n} x(i) \to y(i_2) - y(i_1) = \sum_{i_1+1}^{i_2} x(i)$ are independent rvs.

THM: $y(n)$ II$\to$ $E[y(n)] = \mu n$ and $K_y(i,j) = \sigma^2 \min[i,j]$ for constants μ, σ^2.

Proof: Apply formulae for LTI systems to $h(n) = 1$ for $n \geq 0$; 0 for $n < 0$:
$E[y(n)] = \sum_{i=0}^{n-1} 1 \cdot E[x(n-i)] = n\mu$ where $\mu = E[x(n)]$.
$K_y(m,n) = \sum_{i=0}^{m-1}\sum_{j=0} n - 11 \cdot 1 \cdot \sigma^2\delta(i-j) = \sigma^2 \min[m,n]$.
Note that an II process is *not* WSS since $K_y(m,n) \neq K_y(m-n)$.