Given: pdf $f_x(X)$ and function $y = g(x)$. **Goal:** Compute pdf $f_y(Y)$.

pmfs: $p_y(Y) = Pr[y = Y] = Pr[g(x) = Y] = Pr[x \in g^{-1}(Y)] = \sum_{g^{-1}(Y)} p_x(X)$.

Scale: $y = ax \rightarrow f_y(Y) = \frac{1}{|a|} f_x(\frac{Y}{a})$ so integrates to one. Compare to $\delta(X)$.

Shift: $y = x - b \rightarrow f_y(Y) = f_x(Y + b)$ just shift pdf.

I. Method of events: Straightforward; $g(\cdot)$ need not be differentiable.

1. $F_y(Y) = Pr[y \leq Y] = Pr[g(x) \leq Y] = Pr[x \in g^{-1}(\{y : y \leq Y\})]$.
2. $f_y(Y) = \frac{d}{dY} F_y(Y) = \frac{d}{dY} \int_{X \in g^{-1}(\{y: y \leq Y\})} f_x(X) dX$.

EX1: $f_x(X) = \frac{1}{2}, 0 < X < 2$; 0 otherwise. $y = g(x) = 1/x$. Compute $f_y(Y)$.

1. $F_y(Y) = Pr[y \leq Y] = Pr[\frac{1}{x} \leq Y] = Pr[x \geq \frac{1}{Y}]$. $X < 2 \rightarrow Y > \frac{1}{2}$.
2. $F_y(Y) = \begin{cases} \frac{1}{2}(2 - \frac{1}{Y}) & \text{if } Y > \frac{1}{2} \\ 0 & \text{if } Y < \frac{1}{2} \end{cases} \rightarrow f_y(Y) = \begin{cases} 1/(2Y^2) & \text{if } Y > \frac{1}{2} \\ 0 & \text{if } Y < \frac{1}{2} \end{cases}$.
3. **Check:** $F_y(Y)$ is continuous at $Y = \frac{1}{2}$; $F_y(-\infty) = 0$; $F_y(\infty) = 1$.

EX2: Arbitrary $f_x(X)$. $y = |x|$. Compute $f_y(Y)$ in terms of $f_x(X)$.

1. $F_y(Y) = Pr[y \leq Y] = Pr[|x| \leq Y] = Pr[-Y \leq x \leq Y] = \int_{-Y}^{Y} f_x(X) dX$.
2. $f_y(Y) = \frac{d}{dY} \int_{-Y}^{Y} f_x(X) dX = f_x(Y) + f_x(-Y), Y \geq 0$; 0 if $Y < 0$.

II. Jacobian method: Requires $g(\cdot)$ to be differentiable, but easier.

1. Suppose $g(\cdot)$ is any nondecreasing ($\rightarrow$1-1) and differentiable function.

2a. $Pr[a < x < b] = Pr[g(a) < y < g(b)] = \int_{g(a)}^{g(b)} f_y(Y) dY$.

2b. $Pr[a < x < b] = \int_a^b f_x(X) dX = \int_{g(a)}^{g(b)} f_x(X = g^{-1}(Y)) |dx/dy| dY$.

2c. These are equal if $f_y(Y) = f_x(X = g^{-1}(Y)) |dg^{-1}(Y)/dY|$.

EX1: $y = 1/x \rightarrow X = g^{-1}(Y) = 1/Y$ is *decreasing* and 1-1 for $X, Y > 0$.

$$f_y(Y) = |\frac{d(1/Y)}{dY}| \begin{cases} \frac{1}{2} & \text{if } Y > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/(2Y^2) & \text{if } Y > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

EX2: Can't use Jacobian since $y = |x|$ not differentiable at $x = 0$.

Given: $f_{x,y}(X,Y) = 1$ for $0 \leq X, Y \leq 1$; 0 otherwise (uniform 2-D pdf).
Given: $z = min(x,y)$ and $w = max(x,y)$. **Goal:** Compute $f_{z,w}(Z,W)$.
How? Must use method of events, since min and max non-differentiable.

$$F_{z,w}(Z,W) = Pr[z \leq Z \,\&\, w \leq W] = Pr[min(x,y) \leq Z \,\&\, max(x,y) \leq W].$$

Must consider various cases of Z, W separately below:

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1. $0 \leq W \leq Z$: $F_{z,w}(Z,W) = Pr[x \leq W \,\&\, y \leq W] = W^2$.

2a. $0 \leq Z \leq W \leq 1:\quad F_{z,w}(Z,W) =$
$Pr[(x \leq W \,\&\, y \leq W) \cap (x \leq Z \; or \; y \leq Z)] = 2ZW - Z^2$.

2b. $0 \leq Z \leq W; Z \leq 1; W \geq 1:\quad F_{z,w}(Z,W) =$
$Pr[min(x,y) \leq Z] = 1 - Pr[x \geq Z \,\&\, y \geq Z] = 1 - (1-Z)^2$.

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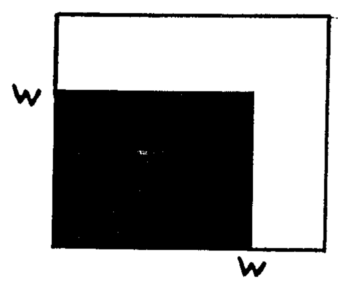
3. $Z, W \leq 0:\quad F_{z,w}(Z,W) = 0.\quad Z, W \geq 1:\quad F_{z,w}(Z,W) = 1.$

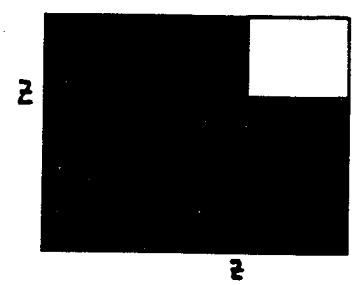
$$f_{z,w}(Z,W) = \frac{\partial^2}{\partial Z \partial W} F_{z,w}(Z,W) = \begin{cases} 2 & \text{if } 0 \leq Z \leq W \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Note: $\int\int f_{z,w}(Z,W) dZ\, dW = 2 \cdot Area[triangle] = 1$. Checks.

CASE 1



CASE 2b



CASE 2a