DEF: A set is *finite* if it has a finite number of elements.
DEF: Two sets A, B are in *one-to-one correspondence* ("1-1") if there exists a 1-1 mapping between elements of A and elements of B.
NOTE: Two *finite* sets are 1-1 IFF they have same number of elements.
EX: $\{a, b, c \ldots z\}$ and $\{101, 102 \ldots 126\}$ are 1-1 (26 elements each).

NOTE: An *infinite* set can be 1-1 with a *proper* subset of itself:
$A = \{1, 2, 3, 4 \ldots\}$ and $B = \{2, 4, 6, 8 \ldots\}$ are 1-1: Mapping $b = 2a$.
$Z = \{\ldots -2, -1, 0, 1, 2 \ldots\}$ and $Y = Z^+ = \{1, 2, 3 \ldots\}$ are 1-1:
1-1 Mapping: $z = y/2$ if y is even; $z = (1-y)/2$ if y is odd.

DEF: A set is *countably infinite* IFF it is 1-1 with $\{integers\}$.
i.e.: You can "count" the elements of the set (this may take forever!).
EX: $\{even\, integers\}$ and $\{odd\, integers\}$ are countably infinite.
DEF: A set is *countable* IFF it is either finite or countably infinite.
NOTE: A set is countable IFF it is 1-1 with another countable set.

THM: The set of *lattice points* $\mathcal{Z}^2 = \{(i, j) : i, j \in \{integers\}\}$ is countable.
Proof: A 1-1 mapping between $\{(i, j) : i = 0, 1, 2 \ldots, j = 0, 1, 2 \ldots\}$ and $\{n : n = 1, 2, 3 \ldots\}$ is $n = (i + j + 1)(i + j + 2)/2 - j$.
Can easily extend to negative values as shown above.

THM: The set of *Rationals* $\mathcal{Q} = \{i/j : i, j \in \{integers\}; j > 0\}$ is countable.
Proof: 1-1 Mapping: $\mathcal{Q} \ni q = i/j \leftrightarrow (i, j) \in \mathcal{Z}^2$ is known to be countable.
In fact, $\mathcal{Q}$ is 1-1 with a *subset* of $\mathcal{Z}^2$; $\mathcal{Q}$ is *at most* countable!
BUT: Countable $\mathcal{Z} \subset \mathcal{Q}$, so $\mathcal{Q}$ is *at least* countable $\rightarrow$ $\mathcal{Q}$ is *exactly* countable.

THM: A countable union of countable sets is countable.
Proof: The "countable union" is 1-1 with $\mathcal{Z}^+$; reindex it with $i = 1, 2 \ldots$
The "countable sets" can similarly *each* be reindexed with $j = 1, 2 \ldots$
$\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} \{a_{i1}, a_{i2} \ldots\} = \cup_{i=1}^{\infty} \cup_{j=1}^{\infty} \{a_{i,j}\} \leftrightarrow (i, j) \in \mathcal{Z}^2$.
Again, possible duplications$\rightarrow$this is *at most* countable.
This theorem is particularly useful for showing that a set is countable.

DEF: An *uncountable* set is NOT 1-1 with *any* countable set.
THM: (Cantor 1890) $[0, 1)$ (Ω for the wheel of fortune) is an *uncountable* set!
Proof: Suppose $[0, 1)$ is countable. Index all $x \in [0, 1)$ as $\{x_1, x_2 \ldots\}$.
Let x_n have *binary* expansion $x_n = 0.x_{n1}x_{n2}x_{n3}...$ where $x_{nj} = 0$ or 1.
Let $y_{nj} = 1 - x_{nj}$. Then $y = 0.y_{11}y_{22}y_{33} \ldots \neq x_n$ for all n!

- Since $\Omega = [0, 1)$ for the wheel of fortune is uncountable, the third axiom of probability does not hold in the $0 = 1$ "proof."

THM: The *power set* of a countably infinite set A is uncountable.
Proof: Index A as $A = \{a_1, a_2 \ldots\}$. Let $B \in \mathcal{P}(A)$=power set of A. Then each B can be indexed by $\{b_1, b_2 \ldots\}$ where $b_n = 1$ if $a_n \in B$ and $b_n = 0$ if $a_n \notin B$. The set of all possible such strings of 0's and 1's is 1-1 with $[0, 1)$, represented using a binary expansion.

THM: The set of real numbers $\mathcal{R}$ is uncountable and 1-1 with $(0, 1)$.
Proof: $\mathcal{R}$ is 1-1 with $(0, 1)$ using $r = \tan(\pi(x - 1/2))$ where $x \in (0, 1)$.
Note: A useful tool: show a set is 1-1 with a set known to be countable.
Note: Omitting fine print: repeating decimals in $[0, 1)$; i/j lowest terms.

Fact: A countable *product* of countable sets need *not* be countable.
Proof: $\prod_{n=1}^{\infty}\{0, 1\}$ is 1-1 with $[0, 1)$, represented using a binary expansion.

Note: If sample space Ω is *finite*, can use $\mathcal{A} = \mathcal{P}(\Omega)$ as event space. If Ω is infinite (countable or not), must *generate* event space $\mathcal{A}$ using some *subset* of Ω to which probabilities can be assigned.
EX: For the wheel of fortune, $\Omega = [0, 1)$ is uncountable and $\mathcal{P}([0, 1)) = \aleph_2$. So *generate* event space $\mathcal{A}$ from all intervals $(a, b), 0 \leq a \leq b \leq 1$, since *assign* $Pr[(a, b)] = b - a$ and *compute* $Pr[B]$ for any $B \in \mathcal{B}$.
DEF: *Probability space* $(\Omega, \mathcal{A}, Pr : \mathcal{A} \to [0, 1])$. Here $\mathcal{A} = \mathcal{B}$=Borel sets. Any subset of $[0, 1)$ or $\mathcal{R}$ you are likely to encounter is a Borel set.

REVIEW OF MAPPINGS AND FUNCTIONS

DEF: A *mapping* or *function* $f : D \to R$ from *domain* D to *range* R assigns to each $d \in D$ a unique $r \in R$, where $r = f(d)$.
DEF: $f : D \to R$ is *onto* IFF $\forall r \in R, \exists d \in D$ s.t. $f(d) = r$.
DEF: $f : D \to R$ is *into* IFF $\exists r \in R$ s.t. $f(d) \neq r \quad \forall d \in D$.
DEF: $f : D \to R$ is *one-to-one* ("1-1") IFF $\forall r \in R, \exists! d \in D$ s.t. $f(d) = r$.
$\forall$=for all; $\exists$=there exists; s.t.=such that; iff=if and only if.

THM: A mapping that is 1-1 and onto is *invertible*: $\exists f^{-1} : R \to D$.
DEF: The *image* of $A \subset D$ is $f(A) = \{b \in R : b = f(a)\text{ for some } a \in A\}$.
DEF: The *preimage* of $B \subset R$ is $f^{-1}(B) = \{a \in D : f(a) \in B\}$.
THM: $A \subset f^{-1}(f(A))$ since $(x \in A) \to (f(x) \in f(A)) \to x \in f^{-1}(f(A))$.

EX: $f(x) = x^2$. $f : \mathcal{R} \to \mathcal{R}$ is *into*; $f : \mathcal{R} \to \{x : x \geq 0\}$ is *onto*.
$f^{-1}(f([2, 3])) = f^{-1}([4, 9]) = [2, 3] \cup [-3, -2] \supset [2, 3]$ for both.
$f : \{x : x \geq 0\} \to \{x : x \geq 0\}$ is 1-1 and onto and thus invertible.

DEF: *Product space* $A \times B = \{(a, b) : a \in A, b \in B\}$. $\mathcal{R}^N = \{N - vectors\}$.
Watch: $[0, 1]^3$=unit cube vs. $\{0, 1\}^3$=8 lattice points. $A - B = A \cap B'$.