DEF: A continuous-time random process \( x(t) \) is a mapping \( x : \Omega \to \mathcal{R}^\mathcal{R} \), or: \( x(t, \omega) : (\mathcal{R} \times \Omega) \to \mathcal{R} \) where \( \Omega \) = sample space and \( \mathcal{R} = \{ \text{reals} \} \).

1. Fix \( t_o \in \mathcal{R} \to x(t_o, \omega) = \) random variable indexed by time index \( t_o \).
2. Fix \( \omega_o \in \Omega \to x(t, \omega_o) = \) sample function = realization (not continuous).
3. Kolmogorov Extension Thm no longer holds: \( \exists \) unmeasurable rps.

DEF: \( x(t) \) is \( N^{th} \)-order stationary if joint pdfs of order \( N \) have:

\[
 f_{x(t_1)\ldots x(t_N)}(X_1 \ldots X_N) = f_{x(t_1+\tau)\ldots x(t_N+\tau)}(X_1 \ldots X_N) \text{ for any } \tau.
\]

DEF: \( x(t) \) SSS strict sense stationary \( \Leftrightarrow \) \( N^{th} \)-order stationary for all \( N \).

Note: iid \( \Rightarrow \) SSS \( \Rightarrow \) \( N^{th} \)-order \( \Rightarrow \) WSS \( \Rightarrow \) \( 1^{st} \)-order \( \Rightarrow \) id.

DEF: \( x(t) \) Gaussian \( \Leftrightarrow \{ x(t_1), x(t_2) \ldots x(t_N) \} \) JGRV for all \( t_1 \ldots t_N \).

\[
 E[y(t)] = \int_{-\infty}^{\infty} h(t-s)E[x(s)]ds = \int_{-\infty}^{\infty} h(s)E[x(t-s)]ds \nonumber \text{ (} h(t) \text{ = impulse response)}.
\]

\[
 K_{x_y}(t,s) = \int_{-\infty}^{\infty} h(u)K_x(t,s-u)du = \int_{-\infty}^{\infty} h(s-u)K_x(t,u)du = K_{yx}(s,t).
\]

\[
 K_y(t,s) = \int \int h(t-u)h(s-v)K_x(u,v)du dv = \int \int h(u)h(v)K_x(t-u,s-v)du dv.
\]

\[
 WSS: \quad E[y(t)] = \mu \int_{-\infty}^{\infty} h(s)ds = H(0)\mu = \text{constant}.
\]

\[
 WSS: \quad K_y(t,s) = \int \int h(u)h(v)K_x((t-u)-(s-v))du dv = \int \int h(u)h(v)K_x((t-s)-u+v)du dv = K_y(t-s) \Rightarrow y(t) \text{ also WSS}.
\]

\[
 WSS: \quad K_{xy}(t,s) = \int h(u)K_x(t-s+u)du = K_{xy}(t-s) \Rightarrow x,y \text{ jointly WSS}.
\]

Transfer function: \( H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \Leftrightarrow h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)e^{j\omega t}d\omega. \)

PSD: \( S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega \tau}d\tau = 2 \int_{0}^{\infty} R_x(\tau)\cos(\omega \tau)d\tau \quad (R_x(\tau) \text{ even}). \)

Then: \( S_y(\omega) = H(\omega)H(-j\omega)S_x(\omega) = |H(\omega)|^2 S_x(\omega). \) Much more later!

EX1: Wiener process=0-mean, II, Gaussian process \( \Leftrightarrow \) cont-time random walk.

EX2: Poisson counting process=II; Poisson increments \( \Leftrightarrow \) # arrivals in [0, \( t \)].

EX3: WGN (White Gassian Noise)= white Gaussian rp (need not be noise).

EX4: WGN \( \Rightarrow \) \( |\frac{dy}{dt} + ay = bu| \) \( \Rightarrow \) Gauss-Markov rp (1\textsuperscript{st}-order system only).