Three Card Monte There are 3 cards: red/red; red/black; black/black.
The cards are shuffled and one chosen at random.
The top of the card is red. Pr[bottom is red]=?

---

3 possible lines of reasoning to solve this problem:

1. Bottom is red only if chose red/red card→ $Pr = 1/3$.
2. Not black/black, so either red/black or red/red→ $Pr = 1/2$.
3. 5 hidden sides: 2 red and 3 black→ $Pr = 2/5$.

Which is correct? They are ALL wrong! In fact, $Pr = 2/3$.

DEF: $Pr[A|B] = Pr[$event A occurs, GIVEN THAT event B occurrED$]$.

- Either A occurs or A doesn't occur, even if B occurred.
- Their *relative* probabilities (ratio) shouldn't change, after restriction to B occurring ($A \cap B$ and $A' \cap B$).

Want: $Pr[A|B] + Pr[A'|B] = 1$ and $\frac{Pr[A|B]}{Pr[A'|B]} = \frac{Pr[A\cap B]}{Pr[A'\cap B]}$.

Know: $Pr[A \cap B] + Pr[A' \cap B] = Pr[B]$. So just divide this by $Pr[B]$.

THM: $Pr[A|B] = Pr[A \cap B]/Pr[B] = Pr[A \cap B]/(Pr[A \cap B] + Pr[A' \cap B])$.

NOTE: Forms: $Pr[A|B] = \frac{x}{x+y}$ and $Pr[A'|B] = \frac{y}{x+y}$. Ratio x/y, add to one.

EX: $Pr[Bottom\,red|Top\,red] = \frac{Pr[Top\ AND\ Bottom\,red]}{Pr[Top\,red]} = \frac{1/3}{1/2} = 2/3$.

OR: $Pr[BR|TR] = \frac{Pr[RR]}{Pr[RR]Pr[TR|RR]+Pr[RB]Pr[TR|RB]+Pr[BB]Pr[TR|BB]}$
$= \frac{1/3}{(1/3)(1)+(1/3)(1/2)+(1/3)(0)} = 2/3$ (try drawing a 3-branch tree).

Why? BB eliminated; 1 red face up; 2 of 3 remaining faces are red!

Bayes's $Pr[A|B] = \frac{Pr[A\cap B]}{Pr[B]} = Pr[B|A]\frac{Pr[A]}{Pr[B]} = \frac{Pr[B|A]Pr[A]}{Pr[B|A]Pr[A]+Pr[B|A']Pr[A']}$.

Why? Suppose we are given $Pr[A]$ (*a priori* prob. of A occurring).
Now we observe B occurs. How does this *change* prob. of A?

i.e.: Compute *a posteriori* prob. $Pr[A|B]$ of A occurring, given B.

EX: Coin A has Pr[heads]=1/3; Coin B has Pr[heads]=3/4.
Choose coin at random, flip it, get heads. Compute Pr[coin A].

Note: Without observing the flip result, Pr[coin A]=1/2 (*a priori*).

But: $Pr[A|H] = \frac{Pr[H|A]Pr[A]}{Pr[H|A]Pr[A]+Pr[H|B]Pr[B]} = \frac{(1/3)(1/2)}{(1/3)(1/2)+(3/4)(1/2)} = \frac{4}{13} < \frac{1}{2}$.

Why? Observed heads→more likely chose coin more likely to land heads.

DEF: *Increasing sequence* of sets $A_1 \subset A_2 \subset A_3 \subset \ldots$ $\lim_{n\to\infty} A_n = \cup_{n=1}^{\infty} A_n$.

DEF: *Decreasing sequence* of sets $A_1 \supset A_2 \supset A_3 \supset \ldots$ $\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n$.

Thm: $Pr[\lim_{n\to\infty} A_n] = \lim_{n\to\infty} Pr[A_n]$ for either increasing or decreasing sets.

Note: We can interchange "limit" and the function "Pr"; Pr is *continuous*.

Proof: For the increasing sequence $\{A_n\}$, let $B_n = A_n - A_{n-1}, A_0 = \phi$.
$Pr[A_n] = Pr[\cup_{i=1}^{n} A_i] = Pr[\cup_{i=1}^{n} B_i] = \sum_{i=1}^{n} Pr[B_i] \quad (B_i \cap B_j = \phi)$.

lim: $\lim_{n\to\infty} Pr[A_n] = \sum_{i=1}^{\infty} Pr[B_i] = Pr[\cup_{i=1}^{\infty} B_i] = Pr[\lim_{n\to\infty} A_n]$
using the third axiom for countably infinite union of disjoint B_i.

- The proof for a decreasing sequence of sets $\{A_n\}$ is similar.

Note: If we *start* with disjoint $\{B_i\}$ and define $A_n = \cup_{i=1}^{n} B_i$,
and we *suppose* that continuity of probability is true,
we can use this argument to *derive* the third axiom!
Historically, this is the way Kolmogorov did it in 1933.

DEF: For *any* sequence of sets $\{A_n\}$, we can define the *limsup* and *liminf*:

$$\limsup_{n\to\infty} A_n = \lim_{n\to\infty} \cup_{i=n}^{\infty} A_i; \quad \liminf_{n\to\infty} A_n = \lim_{n\to\infty} \cap_{i=n}^{\infty} A_i.$$

Apply continuity of probability using limsup and liminf, not lim.

- Property: $(\limsup_{n\to\infty} A_n)' = \liminf_{n\to\infty} A_n'$; $\quad (\liminf_{n\to\infty} A_n)' = \limsup_{n\to\infty} A_n'$.

EX1: Spin a wheel of fortune. Compute $Pr[\{\frac{1}{2}\}]$ using cont. of probability.

$$Pr[\{\tfrac{1}{2}\}] = Pr[\lim_{n\to\infty}(\tfrac{1}{2} - \tfrac{1}{n}, \tfrac{1}{2} + \tfrac{1}{n})] = \lim_{n\to\infty} Pr[(\tfrac{1}{2} - \tfrac{1}{n}, \tfrac{1}{2} + \tfrac{1}{n})] = \lim_{n\to\infty} \tfrac{2}{n} = 0$$

since $A_n = (\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n})$ is a decreasing sequence of sets.
We already knew this, but now we can use this simpler derivation.

EX2: Spin a wheel. Compute Pr[decimal expansion WON'T contain a 6].
Let $A = \{x : 0 \le x < 1\}$ and the decimal expansion of x has no 6.

$$A = [0,1) - [0.6, 0.7) - \cup_{\substack{n=0\\n\neq 6}}^{9} [0.n6, 0.n7) - \cup_{\substack{i=0\\i\neq 6}}^{9} \cup_{\substack{j=0\\j\neq 6}}^{9} [0.ij6, 0.ij7) - \ldots$$

$$A_{n+1} = A_n - B_n = A_n - \cup_{\substack{i_1=0\\i_1\neq 6}}^{9} \cdots \cup_{\substack{i_n=0\\i_n\neq 6}}^{9} [0.i_1 \ldots i_n 6, 0.i_1 \ldots i_n 7).$$

$B_n \subset A_n \to Pr[A_n - B_n] = Pr[A_n] - Pr[B_n]$. Decreasing sequence.

Cont. of prob.$\to Pr[A] = \lim_{n\to\infty} Pr[A_n] = \lim_{n\to\infty}(1 - \sum_{i=0}^{n-1} Pr[B_i])$
$= \lim_{n\to\infty}(1 - 0.1 - 9(0.1)^2 - 9^2(0.1)^3 - \ldots - 9^{n-1}(0.1)^n) = \lim_{n\to\infty}(0.9)^n = 0.$

Note: Heuristically, Pr[none of 1st n digits are 6]=$(0.9)^n$. Indpt. digits?