DEF: rvs $\{x_1, x_2 \ldots\}$ are *iidrv* with means μ and variances σ^2 if:

1. The x_i are *independent*: $f_{x_1,x_2\ldots}(X_1, X_2 \ldots) = f_{x_1}(X_1) f_{x_2}(X_2) \ldots$
2. x_i are *identically distributed*: $f_{x_i}(X) = f_x(X), E[x_i] = \mu, \sigma^2_{x_i} = \sigma^2$.

THM: $y_n = \sum_{i=1}^n x_i$=sum of iidrvs x_i with *finite* means μ and variance σ^2.
Then: $E[y_n] = n\mu; \sigma^2_{y_n} = n\sigma^2; \tilde{y}_n = \frac{y_n - n\mu}{\sqrt{n}\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{x_i - \mu}{\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{x}_i$.

Mean: Let $m_n = \frac{y_n}{n} = \frac{1}{n} \sum_{i=1}^n x_i = mean$. Then $E[m_n] = \mu$ and $\sigma^2_{m_n} = \frac{\sigma^2}{n}$.
Proof: All of these follow immediately from the basic properties of variance.
Note: Variance of (sample) mean gets *smaller* with n! "Regression to mean." While variance of y_n grows as n, variance of $\frac{y_n}{n}$ "grows" as $\frac{1}{n} \to 0$.
Note: Does *not* mean that m_n "remembers" to correct deviations from μ!

DEF: The *characteristic function* $\Phi_x(\omega)$ of rv x is
$\Phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} f_x(X) e^{j\omega X} dX = \mathcal{F}\{f_x(X)\}$ (note sign).

FACT: Let x, y be independent rvs and $z = x + y$. Then $f_z(Z) = f_x(Z) * f_y(Z) = \int f_x(X) f_y(Z - X) dX$ and $\Phi_{x+y}(\omega) = \Phi_x(\omega)\Phi_y(\omega)$.
Proof: See recitation notes and text p. 127,135,152. $\Phi_x(\omega)$: see p. 204-209.
Or: $\Phi_{x+y}(\omega) = E[e^{j\omega(x+y)}] = E[e^{j\omega x}]E[e^{j\omega y}] = \Phi_x(\omega)\Phi_y(\omega)$ QED.

THM: Basic form of Central Limit Theorem (CLT):
Let $x_1, x_2 \ldots$ be iidrvs with *finite* means μ and variances σ^2.
Then the *normalized* $\tilde{y}_n = (\sum_{i=1}^n x_i - n\mu)/(\sqrt{n}\sigma) \to r$ in distribution,
where r is a *unit Gaussian* rv with pdf $f_r(R) = \frac{1}{\sqrt{2\pi}} e^{-R^2/2}$ $(\sigma_r^2 = 1)$

1. "Convergence in distribution" means $F_{\tilde{y}_n}(Y) \to F_r(R)$ pointwise.
2. Note that a binomial pmf *cannot* converge to a Gaussian pdf, but a binomial PDF can converge to a Gaussian PDF (distributions).

Proof: Essentially a Taylor series expansion of the characteristic function.

$\Phi_{\tilde{y}_n}(\omega) = E[e^{j\omega\tilde{y}_n}] = E[e^{j\omega\tilde{x}_1/\sqrt{n}}] \ldots E[e^{j\omega\tilde{x}_n/\sqrt{n}}] = (E[e^{j\omega\tilde{x}/\sqrt{n}}])^n$
$= (E[1 + j\omega\frac{\tilde{x}}{\sqrt{n}} - \frac{\omega^2}{2}\frac{\tilde{x}^2}{n} + \ldots])^n = (1 + \frac{j\omega}{\sqrt{n}} E[\tilde{x}] - \frac{\omega^2}{2n} E[\tilde{x}^2] + \ldots)^n$
$= (1 - \omega^2/(2n) + H.O.T.)^n \simeq e^{-\omega^2/2} = \Phi_r(\omega)$ as $n \to \infty$.
$\Phi_{\tilde{y}_n}(\omega) \to \Phi_r(\omega)$ pointwise$\to$convergence in distribution. QED.

H.O.T. =Higher Order Terms. *Normalized* $\tilde{x} = \frac{x - E[x]}{\sigma_x} \to E[\tilde{x}] = 0, \quad \sigma^2_{\tilde{x}} = 1$.

DEF: $\Phi(X) = \int_{-\infty}^{X} \frac{1}{\sqrt{2\pi}} e^{-R^2/2} dR$=PDF for unit (normalized) Gaussian.

Note: $erf(X) = \int_{0}^{X} \frac{1}{\sqrt{2\pi}} e^{-R^2/2} dR$ or $\int_{-X}^{X} \frac{1}{\sqrt{2\pi}} e^{-R^2/2} dR$. See table p. 62.

Note: $\Phi(-X) = 1 - \Phi(X)$ and $\Phi(X) < \frac{1}{2} \rightarrow X < 0$ and $erf(-X) = -erf(X)$.

Procedure for using CLT to compute $Pr[a < y < b]$, where y is sum of n iidrvs x_i with known means $E[x]$ and variances σ_x^2:

1. Compute $E[y] = nE[x]$ and $\sigma_y^2 = n\sigma_x^2$ and $\sigma_y = \sqrt{\sigma_y^2}$. **Square root!**
2. $Pr[a < y < b] = \Phi(\frac{b-E[y]}{\sigma_y}) - \Phi(\frac{a-E[y]}{\sigma_y})$ and $Pr[y < b] = \Phi(\frac{b-E[y]}{\sigma_y})$.
3. Use $\Phi(-X) = 1 - \Phi(X)$ as needed. $\Phi(-\infty) = 0, \Phi(0) = \frac{1}{2}, \Phi(\infty) = 1$.
4. If y is a *discrete* rv which takes on integer values (including a,b), use the *Demoivre-Laplace correction* to the central limit theorem: $Pr[a \leq y \leq b] = \Phi((b + \frac{1}{2} - E[y])/\sigma_y) - \Phi((a - \frac{1}{2} - E[y])/\sigma_y)$.

EX1: $f_x(X)$ is Gaussian pdf. Sum of independent Gaussian rvs is Gaussian.

EX2: $f_x(X) = 1/\pi(1 + X^2)$ (Cauchy pdf)$\rightarrow E[x] = 0$ (Cauchy prin. value).

But: $\sigma_x^2 = E[x^2] = \int_{-\infty}^{\infty} X^2/\pi(X^2+1)dX \rightarrow \infty$ so CLT *does not apply.*

EX3: $p_x(0) = p_x(1) = \frac{1}{2} \rightarrow p_{y_n}(Y) = \binom{n}{Y}(\frac{1}{2})^n \simeq \frac{1}{\sqrt{2\pi n/4}} e^{-(Y-\frac{n}{2})^2/(2n/4)}$.

EX4: Demoivre-Laplace correction: Flip a fair coin 100 times (indpt flips).
Pr[55 heads]=$\binom{100}{55}(\frac{1}{2})^{100} = 0.0485$. $E[y] = 50; \sigma_y^2 = 100\frac{1}{2}\frac{1}{2} = 25$.
Pr[55 heads]=$Pr[55 \leq y \leq 55] = \Phi(\frac{55.5-50}{\sqrt{25}}) - \Phi(\frac{54.5-50}{\sqrt{25}}) = 0.0484$

610: Exact answer: $\int_{16}^{22}(5^{100}S^{99}e^{-5S})/99!\, dS$ (100^{th}-order Erlang pdf).

CLT: $E[s] = 100E[x] = 100(\frac{1}{5}) = 20$. $\sigma_s^2 = 100\sigma_x^2 = 100(\frac{1}{5})^2 = 4$. $\sigma_s = 2$.
$Pr[16 < s < 22] = \Phi(\frac{22-20}{2}) - \Phi(\frac{16-20}{2}) = \Phi(1) - \Phi(-2) = 0.8185$.

(b): $Pr[|s - E[s]| > 2\sigma_s] = 2Pr[(s - E[s])/\sigma_s > 2] = 2(1 - \Phi(2)) = 0.0456$.
Chebyschev$\neq$: $Pr[|s - E[s]| > 2\sigma_s] \leq \sigma_s^2/(2\sigma_s)^2 = 0.25$. Very loose!

612: $E[s] = 1680E[x] = 5880$. $\sigma_s^2 = 1680\frac{35}{12} = 4900$. $\sigma_s = 70$.
$Pr[s > 5600] = 1 - \Phi((5600 - 5880)/70) = 1 - \Phi(-4) = \Phi(4) = 0.9999$.

(b): $0.99 = Pr[|s - E[s]| < K] = Pr[-\frac{K}{\sigma_s} < \frac{s-E[s]}{\sigma_s} < \frac{K}{\sigma_s}] = \Phi(\frac{K}{\sigma_s}) - \Phi(-\frac{K}{\sigma_s}) = 2\Phi(\frac{K}{\sigma_s}) - 1 \rightarrow \Phi(K/70) = 0.995 \rightarrow K = 70(2.58) = 181$.