Recitation 7 - EECS 451, Winter 2010

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OUTLINE

- Review of Discrete Fourier Transform
- Practice problems

Concepts: Discrete Fourier Tranform

1) For a time limited signal $x(n)$ of length N, the N-point DFT is defined by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

where $k = 0, 1, ..., N-1$ and the inverse DFT is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}}$$

where $n = 0, 1, ..., N-1$.

- Note that the indices range from 0 to $N-1$ for $x(n)$ and $X(k)$
- The DFT is efficiently implemented by the Fast Fourier transform (FFT)

2) Relationship to other transformations

a) If $x_p(n)$ is the N-periodic superposition of $x(n)$, then the DTFS of $x_p(n)$ is obtained by

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi kn}{N}}$$

thus leading to $X(k) = N c_k$.

b) The DFT of $x(n)$ has the following relationship with $X(e^{j\omega})$ and $X(z)$.

$$X(k) = X(e^{j\omega})\big|_{\omega=\frac{2\pi k}{N}}, \qquad X(k) = X(z)\big|_{z=e^{j\frac{2\pi k}{N}}}.$$

c) Properties of the DFT

- Cyclic Convolution: $x_1(n)\copyright x_2(n) \rightarrow X_1(k)X_2(k)$
- Parseval's relation: $\sum_{n=0}^{N-1}|x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1}|X(k)|^2$
- Symmetry property

$$x(n) = x_R^e(n) + x_R^o(n) + jx_I^e(n) + jx_I^o(n)$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$X(k) = X_R^e(k) + jX_I^o(k) + jX_I^e(k) + X_R^o(k)$$

Problems

1) Let $x(n)$ be an N-point sequence whose z-transform is $X(z)$. Let $y(n)$ be an $\frac{N}{2}$-point sequence given as follows.

$$y(n) = x(n) + x(n + \frac{N}{2})$$

for $n = 0, 1, ..., \frac{N}{2} - 1$. Let $Y(k)$ be the DFT of $y(n)$. Find the relationship between $X(z)$ and $Y(k)$.

2) Let $x_p(n)$ be a periodic sequence with fundamental period N. Consider the following DFTs:

$$x_p(n) \overset{N-DFT}{\longrightarrow} = X_1(k)$$

$$x_p(n) \overset{3N-DFT}{\longrightarrow} = X_3(k)$$

Find the relationship between $X_1(k)$ and $X_3(k)$.

3) a) Compute the output $y(n)$ of the LTI system with $h(n) = \{\underline{1}, 0, 0, 1\}$ when the input is $x(n) = \{\underline{2}, 1, 0, 2\}$.
   b) We know that $y(n)$ can be found by taking the inverse DTFT of $H(e^{j\omega})X(e^{j\omega})$. What happens if we find the inverse 4-point DFT of $H(k)X(k)$ instead? Is this $y(n)$?
   c) If the answers for (a) and (b) are not the same, how would you use DFT/IDFT to get $y(n)$?

Recitation 17 solutions by Jung Hyun Bae

1.
$$Y(k) = \sum_{n=0}^{\frac{N}{2}-1} y(n)\, e^{-j\frac{2\pi k n}{\frac{N}{2}}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) e^{-j\frac{2\pi k n}{\frac{N}{2}}} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) e^{-j\frac{2\pi k n}{\frac{N}{2}}}$$

(let $n' = n + \frac{N}{2}$ for the second sum)

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) e^{-j\frac{2\pi k n}{\frac{N}{2}}} + \sum_{n'=\frac{N}{2}}^{N-1} x(n')\, e^{-j\frac{2\pi k}{\frac{N}{2}}\left(n'-\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) e^{-j\frac{2\pi k n}{\frac{N}{2}}} + \sum_{n'=\frac{N}{2}}^{N-1} x(n')\, e^{-j\frac{2\pi k n'}{\frac{N}{2}}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{\frac{N}{2}}}$$

$$= \sum_{n=0}^{N-1} x(n) \left(e^{-j\frac{2\pi k}{\frac{N}{2}}}\right)^{n}$$

$$= X(z)\Big|_{z=e^{-j\frac{2\pi k}{\frac{N}{2}}}}$$

2. $$X_3(k) = \sum_{n=0}^{3N-1} x_p(n) e^{-j\frac{2\pi k n}{3N}}$$

$$= \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi(\frac{k}{3})n}{N}} + \sum_{n=N}^{2N-1} x_p(n) e^{-j\frac{2\pi(\frac{k}{3})n}{N}} + \sum_{n=2N}^{3N-1} x_p(n) e^{-j\frac{2\pi(\frac{k}{3})n}{N}}$$

let $n' = n - N$ let $n' = n - 2N$

$$= \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi(\frac{k}{3})n}{N}} + \sum_{n'=0}^{N-1} x_p(n'+N) e^{-j\frac{2\pi(\frac{k}{3})}{N}(n'+N)}$$

$$+ \sum_{n'=0}^{N-1} x_p(n'+2N) e^{-j\frac{2\pi(\frac{k}{3})}{N}(n'+2N)}$$

$$= \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi(\frac{k}{3})n}{N}} + \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi(\frac{k}{3})n}{N}} e^{-\frac{2}{3}\pi k}$$

$$+ \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi(\frac{k}{3})n}{N}} \cdot e^{-\frac{4}{3}\pi k}$$

$$= X_1\left(\frac{k}{3}\right) + X_1\left(\frac{k}{3}\right) + X_1\left(\frac{k}{3}\right) \quad \text{for } k = 3m.$$

$$= 3X_1\left(\frac{k}{3}\right) \quad \text{for } k = 3m.$$

3. a) $y(n) = x(n) * h(n)$

$y(0) = 2$

$y(1) = 1$

$y(2) = 0$

$y(3) = 2+2 = 4$

$y(4) = 1$

$y(5) = 0$

$y(6) = 2$

b) $y'(n) = x(n) ⓒ h(n)$

$y(0) = 2+1 = 3$

$y(1) = 1$

$y(2) = 2$

$y(3) = 2+2 = 4$

c) $x(n) * h(n) = x'(n) ⓒ h'(n)$

where $x'(n) = \{\underline{2}, 1, 0, 2, 0, 0, 0\}$,

$h'(n) = \{\underline{1}, 0, 0, 1, 0, 0, 0\}$.

Hence $y(n)$ can be found by using

7-point DFT/IDFT.