Recitation 6 - EECS 451, Winter 2010
Feb. 24, 2010

OUTLINE

• Review of important concepts (Lecture 11,12)
• Practice problems

Concepts: Frequency domain analysis of LTI systems

1) Pole zero placement and magnitude response

\[ |H(e^{j\omega})| = \frac{|e^{j\omega} - z_1| |e^{j\omega} - z_2| \ldots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| |e^{j\omega} - p_2| \ldots |e^{j\omega} - p_N|} \]

• when \( e^{j\omega} \) is close to a zero, \( |H(e^{j\omega})| \) is small
• when \( e^{j\omega} \) is close to a pole, \( |H(e^{j\omega})| \) is large

2) Filters

a) Notch filter

• Place pairs of complex conjugate zeros on the unit circle
• e.g.

\[ H(z) = \frac{(z - e^{j0.4\pi})(z - e^{-j0.4\pi})}{(z - 0.9e^{j0.4\pi})(z - 0.9e^{-j0.4\pi})} \]

eliminates the component at \( 0.4\pi \)
• Poles placed near zeros to reduce the bandwidth of the notch

b) Low pass filter

• Place poles near low frequencies and zeros near high frequencies
• e.g.

\[ H(z) = \frac{(z - e^{j0.5\pi})(z - e^{-j0.5\pi})(z - e^{j0.75\pi})(z - e^{-j0.75\pi})(z - e^{j\pi})}{(z - 0.6)(z - 0.8e^{j0.25\pi})(z - 0.8e^{-j0.25\pi})(z - 0.8e^{j0.5\pi})(z - 0.8e^{-j0.5\pi})} \]

c) Comb filter

• Think of a notch filter in which the nulls occur periodically
\[ H(z) = \frac{(z - e^{j0.25\pi})(z - e^{-j0.25\pi})(z - e^{j0.5\pi})(z - e^{-j0.5\pi})(z - e^{j0.75\pi})(z - e^{-j0.75\pi})(z - e^{j\pi})}{(z - 0.9e^{j0.25\pi})(z - 0.9e^{-j0.25\pi})(z - 0.9e^{j0.5\pi})(z - 0.9e^{-j0.5\pi})(z - 0.9e^{j0.75\pi})(z - 0.9e^{-j0.75\pi})(z - 0.9e^{j\pi})} \]
\[ = z^7 + z^6 + \ldots + z + 1 \]
\[ = z^7 + 0.9z^6 + 0.9^2z^5 + \ldots + 0.9^9z + 0.9^7 \]

Concepts: Computing continuous-time fourier transforms using DFT

1) DFT:
\[ X_k = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad X(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \]

Continuous-time FT:
\[ X(f) = \int x(t)e^{-j2\pi ft}dt, \quad x(t) = \int X(f)e^{j2\pi ft}dt \]

2) Assume \( x(t) \) is time-limited to \( 0 < t < T \), and \( X(f) \) is band-limited to \( -B/2 < f < B/2 \)

3) Sample \( t : t = n\Delta_t \) for \( 0 \leq n \leq N - 1 \) with \( \Delta_t = 1/B \)
Sample \( f : f = k\Delta_f \) for \( 0 \leq k \leq N - 1 \) with \( \Delta_f = 1/T \)
\[ N = \frac{1}{\Delta_t \Delta_f} = BT \]

4) \[ X(f) \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi fn\Delta_t}, \quad X(k\Delta_f) \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi kn\Delta_f\Delta_t} \]

5) Similarly,
\[ x(t) \approx \sum_{k=0}^{N-1} X(k\Delta_f)e^{j2\pi tk\Delta_f\Delta_f}, \quad x(n\Delta_t) \approx \sum_{k=0}^{N-1} X(k\Delta_f)e^{j2\pi kn\Delta_f\Delta_f} \]

6) Let \( X_k = X(k\Delta_f)/\Delta_t \) and \( x(n) = x(n\Delta_t) \). Then,
\[ X_k \approx \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad x(n) \approx \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \]

Problems
1) An FIR filter is described by the difference equation

\[ y(n) = x(n) - x(n-1) \]

a) Compute its magnitude and phase response
b) Determine its response to the input \( x(n) = \cos(2\pi n) + 3 \sin(\frac{\pi}{3} n) \)

2) Let \( H(z) = \frac{z^2 + 2}{z(z-2)(z+0.5)} \) be the system function of a causal LTI system. Find an all pass filter \( H_1(z) \) and a minimum phase filter \( H_2(z) \) such that \( H(z) = H_1(z) H_2(z) \).

3) Determine the coefficients of the following notch filter if

\[ y(n) + a_1 y(n-1) = x(n) + b_1 x(n-1) + b_2 x(n-2) \]

a) It completely rejects the frequency component at \( \omega = \pi/4 \)
   b) It amplifies a dc signal by 2
1. a) \[ y(n) = x(n) - x(n-1) \]

\[ y(z) = x(z) \left( 1 - z^{-1} \right) \]

\[ H(z) = \frac{y(z)}{x(z)} = 1 - z^{-1} \]

\[ H(e^{jw}) = 1 - e^{-jw} \]

\[ = (1 - \cos w) + j \sin w \]

\[ |H(e^{jw})| = \sqrt{(1 - \cos w)^2 + \sin^2 w} \]

\[ = \sqrt{2 - 2 \cos w} \]

\[ = \sqrt{4 \sin^2 \frac{w}{2}} \]

\[ = 2 \sin \frac{w}{2} \]

\[ \angle H(e^{jw}) = \tan^{-1} \left( \frac{\sin w}{1 - \cos w} \right) \]

\[ = \tan^{-1} \left( \frac{2 \sin \frac{w}{2} \cos \frac{w}{2}}{2 \sin^2 \frac{w}{2}} \right) \]

\[ = \tan^{-1} \left( \cot \frac{w}{2} \right) \]

\[ = \tan^{-1} \left( \tan \left( \frac{\pi - w}{2} \right) \right) = \frac{\pi}{2} - \frac{w}{2} \]
b) $\cos(2\pi n)$ has $\frac{1}{2}$ at $\pm2\pi$

$3\sin\left(\frac{\pi n}{3}\right)$ has $\pm\frac{3}{2}j$ at $\pm\frac{\pi}{3}$.

$H(e^{j2\pi}) = H(e^{-j2\pi}) = 0$.

$H(e^{j\frac{\pi}{3}}) = 1 - e^{-j\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}j = e^{j\frac{\pi}{3}}$

$H(e^{-j\frac{\pi}{3}}) = 1 - e^{j\frac{\pi}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}j = e^{-j\frac{\pi}{3}}$.

$\gamma(n) = -\frac{3}{2}j e^{j\frac{\pi}{3}} e^{j\frac{\pi}{3}n} + \frac{3}{2}j e^{-j\frac{\pi}{3}} e^{-j\frac{\pi}{3}n}$

$= 3\sin\left(\frac{\pi n}{3} + \frac{\pi}{3}\right)$

2. i) All-pass filter

$|H(e^{j\omega})|^2 = H(z)H(z^{-1}) \Big|_{z = e^{j\omega}} = 1$

Let $H(z) = \frac{A(z)}{B(z)}$. Then

$\frac{A(z)}{B(z)} \frac{A(z^{-1})}{B(z^{-1})} = 1$

$\therefore B(z) = A(z^{-1})$

$H(z) = \frac{A(z)}{A(z^{-1})}$.

If there is a pole (zero) at $z_0$, then

there must be a zero (pole) at $z_0^{-1}$. 
11) Minimum phase filter

All poles and zeros are inside of unit-circle.

\[ H(z) = \frac{z^2 + 2}{z(z - 2)(z + 0.5)} \]

Note that \( H_2(z) \) cannot have factors \( z^2 + 2, \ z - 2 \).

Hence \( H_1(z) = \frac{z^2 + 2}{z - 2} \cdot \frac{z - 2}{z^2 + \frac{1}{2}} \)

\[ H_2(z) = \frac{z^2 + \frac{1}{2}}{z - \frac{1}{2}} \cdot \frac{1}{z(z + \frac{1}{2})} = \frac{z^2 + \frac{1}{2}}{z(z^2 - \frac{1}{4})} \]

3. \[ H(z) = \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1}} = \frac{z^2 + b_1 z + b_2}{z^2 + a_1 z} \]

Since it rejects \( \frac{\pi}{4} \), it has zeros at \( e^{\pm \frac{\pi}{4}} \).

Therefore \( z^2 + b_1 z + b_2 = (z - e^{\frac{\pi}{4}})(z - e^{-\frac{\pi}{4}}) \)

\[ = z^2 - 2z \cos \frac{\pi}{4} + 1 = z^2 - \sqrt{2} z + 1 \]

Since it amplifies DC by 2,

\[ H(e^{j0}) = H(1) = \frac{2 - \sqrt{2}}{1 + a_1} = 2 \]

\[ \therefore a_1 = -\frac{\sqrt{2}}{2} \]

\[ y(n) - \frac{\sqrt{2}}{2} y(n-1) = x(n) - \sqrt{2} x(n-1) + x(n-2) \]