Recitation 6 - EECS 451, Winter 2010

Feb. 24, 2010

OUTLINE

- Review of important concepts (Lecture 11,12)
- Practice problems

Concepts: Frequency domain analysis of LTI systems

1) Pole zero placement and magnitude response
    - $$|H(e^{j\omega})| = \frac{|e^{j\omega} - z_1||e^{j\omega} - z_2|...|e^{j\omega} - z_M|}{|e^{j\omega} - p_1||e^{j\omega} - p_2|...|e^{j\omega} - p_N|}$$
    - when $e^{j\omega}$ is close to a zero, $|H(e^{j\omega})|$ is small
    - when $e^{j\omega}$ is close to a pole, $|H(e^{j\omega})|$ is large
2) Filters
    a) Notch filter
        - Place pairs of complex conjugate zeros on the unit circle
        - e.g.
          $$H(z) = \frac{(z - e^{j0.4\pi})(z - e^{-j0.4\pi})}{(z - 0.9e^{j0.4\pi})(z - 0.9e^{-j0.4\pi})}$$
          eliminates the component at 0.4π
        - Poles placed near zeros to reduce the bandwidth of the notch
    b) Low pass filter
        - Place poles near low frequencies and zeros near high frequencies
        - e.g.
          $$H(z) = \frac{(z - e^{j0.5\pi})(z - e^{-j0.5\pi})(z - e^{j0.75\pi})(z - e^{-j0.75\pi})(z - e^{j\pi})}{(z - 0.6)(z - 0.8e^{j0.25\pi})(z - 0.8e^{-j0.25\pi})(z - 0.8e^{j0.5\pi})(z - 0.8e^{-j0.5\pi})}$$
    c) Comb filter
        - Think of a notch filter in which the nulls occur periodically

- e.g.

$$
\begin{aligned}
H(z) &= \frac{(z-e^{j0.25\pi})(z-e^{-j0.25\pi})(z-e^{j0.5\pi})(z-e^{-j0.5\pi})(z-e^{j0.75\pi})(z-e^{-j0.75\pi})(z-e^{j\pi})}{(z-0.9e^{j0.25\pi})(z-0.9e^{-j0.25\pi})(z-0.9e^{j0.5\pi})(z-0.9e^{-j0.5\pi})(z-0.9e^{j0.75\pi})(z-0.9e^{-j0.75\pi})(z-0.9e^{j\pi})} \\
&= \frac{z^7+z^6+...+z+1}{z^7+0.9z^6+0.9^2z^5+...+0.9^6z+0.9^7}
\end{aligned}
$$

Concepts: Computing continuous-time fourier transforms using DFT

1) DFT:

$$X_k = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \qquad X(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

Continuous-time FT:

$$X(f) = \int x(t)e^{-j2\pi ft}dt, \qquad x(t) = \int X(f)e^{j2\pi ft}dt$$

2) Assume $x(t)$ is time-limited to $0 < t < T$, and $X(f)$ is band-limited to $-B/2 < f < B/2$
3) Sample t: $t = n\Delta_t$ for $0 \le n \le N-1$ with $\Delta_t = 1/B$

   Sample f: $f = k\Delta_f$ for $0 \le k \le N-1$ with $\Delta_f = 1/T$

   $N = \frac{1}{\Delta_t \Delta_f} = BT$
4)

$$X(f) \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi fn\Delta_t}\Delta_t,$$

$$X(k\Delta_f) \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi kn\Delta_f\Delta_t}\Delta_t$$

5) Similarly,

$$x(t) \approx \sum_{k=0}^{N-1} X(k\Delta_f)e^{j2\pi tk\Delta_f}\Delta_f,$$

$$x(n\Delta_t) \approx \sum_{k=0}^{N-1} X(k\Delta_f)e^{j2\pi kn\Delta_f\Delta_t}\Delta_f$$

6) Let $X_k = X(k\Delta_f)/\Delta_t$ and $x(n) = x(n\Delta_t)$. Then,

$$X_k \approx \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \qquad x(n) \approx \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

Problems

1) An FIR filter is described by the difference equation

$$y(n) = x(n) - x(n-1)$$

   a) Compute its magnitude and phase response
   b) Determine its response to the input $x(n) = cos(2\pi n) + 3sin(\frac{\pi}{3}n)$

2) Let $H(z) = \frac{z^2+2}{z(z-2)(z+0.5)}$ be the system function of a causal LTI system. Find an all pass filter $H_1(z)$ and a minimum phase filter $H_2(z)$ such that $H(z) = H_1(z)H_2(z)$

   .

3) Determine the coefficients of the following notch filter if

$$y(n) + a_1 y(n-1) = x(n) + b_1 x(n-1) + b_2 x(n-2)$$

   a) It conpletely rejects the frequency component at $\omega = \pi/4$
   b) It amplifies a dc signal by 2

Solutions for Recitation 6 by Jung Hyun Bae

1. a) $y(n) = x(n) - x(n-1)$

$$Y(z) = X(z)(1 - z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1}.$$

$$H(e^{jw}) = 1 - e^{-jw}$$

$$= (1 - \cos w) + j \sin w$$

$$|H(e^{jw})| = \sqrt{(1-\cos w)^2 + \sin^2 w}$$

$$= \sqrt{2 - 2\cos w}$$

$$= \sqrt{4 \sin^2 \frac{w}{2}}$$

$$= 2 \sin \frac{w}{2}$$

$$\angle H(e^{jw}) = \tan^{-1}\left(\frac{\sin w}{1 - \cos w}\right)$$

$$= \tan^{-1}\left(\frac{2 \sin\frac{w}{2} \cos\frac{w}{2}}{2 \sin^2 \frac{w}{2}}\right)$$

$$= \tan^{-1}\left(\cot \frac{w}{2}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{w}{2}\right)\right) = \frac{\pi}{2} - \frac{w}{2}$$

b) $\cos(2\pi n)$ has $\frac{1}{2}$ at $\pm 2\pi$

$3\sin(\frac{\pi}{3}n)$ has $\mp\frac{3}{2}j$ at $\pm\frac{\pi}{3}$.

$$H(e^{j2\pi}) = H(e^{-j2\pi}) = 0.$$

$$H(e^{j\frac{\pi}{3}}) = 1 - e^{-j\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}j = e^{j\frac{\pi}{3}}$$

$$H(e^{-j\frac{\pi}{3}}) = 1 - e^{j\frac{\pi}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}j = e^{-j\frac{\pi}{3}}.$$

$$\therefore\ y(n) = -\frac{3}{2}j e^{j\frac{\pi}{3}} e^{j\frac{\pi}{3}n} + \frac{3}{2}j e^{-j\frac{\pi}{3}} e^{-j\frac{\pi}{3}n}$$

$$= 3\sin(\frac{\pi}{3}n + \frac{\pi}{3})$$

2. i) All-pass filter

$$|H(e^{jw})|^2 = H(z)H(z^{-1})\Big|_{z=e^{jw}} = 1$$

Let $H(z) = \frac{A(z)}{B(z)}$, Then

$$\frac{A(z)}{B(z)} \frac{A(z^{-1})}{B(z^{-1})} = 1$$

$$\therefore\ B(z) = A(z^{-1})$$

$$H(z) = \frac{A(z)}{A(z^{-1})}.$$

If there is a pole (zero) at z_0, then there must be a zero (pole) at z_0^{-1}.

ii) Minimum-phase filter

All poles and zeros are inside of unit-circle.

$$H(z) = \frac{z^2+2}{z(z-2)(z+0.5)}$$

Note that $H_2(z)$ cannot have factors

z^2+2, $z-2$.

Hence $$H_1(z) = \frac{z^2+2}{z-2} \cdot \frac{z-\frac{1}{2}}{z^2+\frac{1}{2}}$$

$$H_2(z) = \frac{z^2+\frac{1}{2}}{z-\frac{1}{2}} \cdot \frac{1}{z(z+\frac{1}{2})} = \frac{z^2+\frac{1}{2}}{z(z^2-\frac{1}{4})}.$$

3. $$H(z) = \frac{1+b_1 z^{-1}+b_2 z^{-2}}{1+a_1 z^{-1}} = \frac{z^2+b_1 z+b_2}{z^2+a_1 z}.$$

Since it rejects $\frac{\pi}{4}$, it has zeros at $e^{\pm j\frac{\pi}{4}}$.

Therefore $$z^2+b_1 z+b_2 = (z-e^{j\frac{\pi}{4}})(z-e^{-j\frac{\pi}{4}})$$

$$= z^2-2z\cos\frac{\pi}{4}+1 = z^2-\sqrt{2}z+1$$

Since it amplifies DC by 2,

$$H(e^{j0}) = H(1) = \frac{2-\sqrt{2}}{1+a_1} = 2$$

$$\therefore a_1 = -\frac{\sqrt{2}}{2}$$

$$y(n) - \frac{\sqrt{2}}{2}y(n-1) = x(n) - \sqrt{2}x(n-1) + x(n-2)$$