Recitation 5 – EECS 451, Winter 2010

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OUTLINE

- Review of important concepts (Lecture 7-10)
- Practice problems

Concepts: LTI systems and z-transforms

- The z-transform of the impulse response h(n) is called the system function [H(z)]
- DT LTI systems described by LCCDE have a rational z-transform, i.e. $H(z) = \dfrac{A(z)}{B(z)}$
- If a signal y(n) is an output of the system when the input signal is x(n), then their z-transforms are related as Y(z)= H(z)X(z)
- Causality/Stability of the system can be determined by the ROC of H(z)

Concepts: 1-sided z-transforms

1. Definition of 1-sided z-transform
   - For a given DT signal x(n), $X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$, where z is complex valued.
   - For causal signals, $X^+(z)$ and X(z) are the same
   - ROC of $X^+(z)$ is always exterior of a circle

2. Properties of 1-sided z-transform: We have $x(n) \xleftrightarrow{z^+} X^+(z)$
   - Almost all properties of the two-sided z-transform carry over to the one-sided z-transform
   - Time delay: $x(n-k) \xleftrightarrow{z^+} z^{-k}[X^+(z) + \sum_{l=1}^{k} x(-l) z^{l}]$
   - Time advance: $x(n+k) \xleftrightarrow{z^+} z^{k}[X^+(z) - \sum_{l=0}^{k-1} x(l) z^{-l}]$

Concepts: Discrete time Fourier transform (DTFT)

1. For a discrete aperiodic signal x(n)

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega, \qquad X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

   - DTFT is periodic with period 2π

2. Relationship of DTFT to the z-transform
   - If the ROC of X(z) includes the unit circle, DTFT is an evaluation of X(z) on the unit circle, i.e. $X(\omega) = X(z)|_{z=e^{j\omega}}$

3. Properties of DTFT: Many follow from the fact $X(\omega) = X(z)|_{z=e^{j\omega}}$
   - Linearity: $a_1x_1(n) + a_2x_2(n) \leftrightarrow a_1X_1(\omega) + a_2X_2(\omega)$
   - Time shifting: $x(n-k) \leftrightarrow e^{-j\omega k}X(\omega)$
   - Time Reversal: $x(-n) \leftrightarrow X(-\omega)$
   - Frequency shifting: $e^{j\omega_0 n} x(n) \leftrightarrow X(\omega - \omega_0)$
   - Convolution: $x_1(n) * x_2(n) \leftrightarrow X_1(\omega)X_2(\omega)$
   - Multiplication: $x_1(n)x_2(n) \leftrightarrow \dfrac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega) X_2(\omega-\lambda) d\lambda$

• Differentiation in the frequency domain: nx(n) ↔ $j\dfrac{dX(\omega)}{d\omega}$

• Conjugation: $x^*(n) \leftrightarrow X^*(-\omega)$

3. Symmetry properties of DTFT: For a discrete signal x(n) with Fourier transform X(ω)
   • If x(n) is real, X(ω) is conjugate symmetric i.e. X*(ω) = X(−ω)
   • If x(n) is real and even, X(ω) is real and even
   • If x(n) is real and odd, X(ω) is imaginary and odd
   • If x(n) is imaginary and even, X(ω) is imaginary and even
   • If x(n) is imaginary and odd, X(ω) is real and odd

4. Parseval's relation
   • The conservation of average power or energy in the time and frequency domains
   • For a discrete time aperiodic signal, we have the total energy conserved

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_1(\omega)|^2 \, d\omega$$

Concepts: Discrete time Fourier series (DTFS)

1. For a discrete periodic signal x(n)

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi \frac{kn}{N}}, \qquad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{kn}{N}}$$

2. Properties of DTFS
   • $c_{-k} = c_{N-k}$ for k>0
   • Parseval's relation

$$\frac{1}{N}\sum_{n=-0}^{N-1} |x(n)|^2 = \sum_{k=-0}^{N-1} |c_k|^2$$

Problems

1. Use the one-sided z-transform to determine the step response for the system y(n) = 0.25y(n−2) + x(n). Also, y(−1) = 0, y(−2) = 1.

2. Determine the impulse response and the step response of the causal system y(n) = y(n − 1) −0.5y(n − 2) + x(n) + x(n − 1). Determine whether the system is stable.

3. Consider the signal x(n) = 2 + 2 cos(πn /4) + cos(πn/2) +0.5cos(3πn/4). Determine its DTFS and evaluate the power of the signal.

4. Express the Fourier transform of the following signals in terms of X(ω), where x(n) ↔X(ω).
   (a) y(n)=x*(-n).

   (b) y(n) = x(n/2) for n even and 0 for n odd.

Recitation 5 solution by Jung Hyun Bae

1. $$Y^+(z) = 0.25\, z^{-2}\left(Y^+(z) + y(-2)\, z^2 + y(-1)\, z \right) + \frac{1}{1-z^{-1}}$$

$$= 0.25\, z^{-2}\left(Y^+(z) + z^2 \right) + \frac{1}{1-z^{-1}}$$

$$(1-0.25\, z^{-2})\, Y^+(z) = 0.25 + \frac{1}{1-z^{-1}}$$

$$Y^+(z) = \frac{0.25}{1-0.25\, z^{-2}} + \frac{1}{(1-z^{-1})(1-0.25\, z^{-2})}$$

$$= \frac{1.25 - 0.25\, z^{-1}}{(1-0.25\, z^{-2})(1-z^{-1})}$$

$$= \frac{1.25\, z^3 - 0.25\, z^2}{(z+0.5)(z-0.5)(z-1)}$$

$$\frac{Y^+(z)}{z} = \frac{1.25\, z^2 - 0.25\, z}{(z+0.5)(z-0.5)(z-1)}$$

$$= \frac{A_1}{z+0.5} + \frac{A_2}{z-0.5} + \frac{A_3}{z-1}$$

$$A_1 = \frac{1.25 \cdot 0.5^2 + 0.25 \cdot 0.5}{-1 \cdot -1.5} = \frac{7}{24}$$

$$A_2 = -\frac{3}{8}$$

$$A_3 = \frac{4}{3}$$

$$\therefore\ y(n) = \left(\frac{7}{24}(-0.5)^n - \frac{3}{8}(0.5)^n + \frac{4}{3} \right) u(n)$$

2. $Y(z) = z^{-1}Y(z) + 0.5z^{-2}Y(z) + X(z) + z^{-1}X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-z^{-1}-0.5z^{-2}} = \frac{z^2+z}{z^2-z-0.5}$$

$$\frac{H(z)}{z} = \frac{z+1}{z^2-z-0.5} = \frac{A_1}{z-\frac{1+\sqrt{3}}{2}} + \frac{A_2}{z-\frac{1-\sqrt{3}}{2}}$$

$$A_1 = \frac{1+\sqrt{3}}{2}, \quad A_2 = \frac{1-\sqrt{3}}{2}$$

$$h(n) = \left(\left(\frac{1+\sqrt{3}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{3}}{2}\right)^{n+1}\right)u(n).$$

Let $y_1(n)$ be the step response. Then,

$$Y_1(z) = H(z)\cdot\frac{z}{z-1} = \frac{z^3+z^2}{(z^2-z-0.5)(z-1)}$$

$$\frac{Y_1(z)}{z} = \frac{z^2+z}{(z^2-z-0.5)(z-1)} = \frac{A_1}{z-\frac{1+\sqrt{3}}{2}} + \frac{A_2}{z-\frac{1-\sqrt{3}}{2}} + \frac{A_3}{z-1}.$$

$$A_1 = \frac{5+3\sqrt{3}}{2},$$

$$A_2 = \frac{5-3\sqrt{3}}{2},$$

$$A_3 = -4$$

$$\therefore y_1(n) = \left(\frac{5+3\sqrt{3}}{2}\left(\frac{1+\sqrt{3}}{2}\right)^n + \frac{5-3\sqrt{3}}{2}\left(\frac{1-\sqrt{3}}{2}\right)^n - 4\right)u(n)$$

Since $\frac{1+\sqrt{3}}{2} > 1$, the system is not stable.

3. $N=8$.

$$x(n) = 2 + e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n} + \frac{1}{2}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{-j\frac{\pi}{2}n} + \frac{1}{4}e^{j\frac{3}{4}\pi n} + \frac{1}{4}e^{-j\frac{3}{4}\pi n}$$

$$= \sum_{k=0}^{7} c_k e^{j\frac{\pi k n}{4}}$$

$$= \sum_{k=-3}^{4} c_k e^{j\frac{\pi k n}{4}}$$

$\therefore$ $c_0 = 2$, $c_1 = 1$, $c_2 = \frac{1}{2}$, $c_3 = \frac{1}{4}$, $c_4 = 0$,

$c_5 = c_{-3} = \frac{1}{4}$, $c_6 = c_{-2} = \frac{1}{2}$, $c_7 = c_{-1} = 1$.

Using Parseval's relation,

$$P_x = \sum_{k=0}^{7} |c_k|^2 = \left(4+1+\frac{1}{4}+\frac{1}{16}+\frac{1}{16}+\frac{1}{4}+1\right)$$

$$= \frac{53}{8}$$

4. (a) Let $X_1(\omega)$ DTFT of $x^*(-n)$.

Then, $$X_1(\omega) = \sum_{n=-\infty}^{\infty} x^*(-n) e^{-j\omega n}$$

$$= \sum_{n'=-\infty}^{\infty} x^*(n') e^{j\omega n'}$$

$$= \sum_{n'=-\infty}^{\infty} \left(x(n') e^{-j\omega n'}\right)^*$$

$$= \left(\sum_{n'=-\infty}^{\infty} x(n') e^{-j\omega n'}\right)^*$$

$$= X^*(\omega)$$

(b) $$Y(\omega) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} y(2k)\, e^{-j\omega 2k}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, e^{-j(2\omega)k}$$

$$= X(2\omega).$$