Recitation 4 - EECS 451, Winter 2010

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OUTLINE

- Review of important concepts (Lecture 5-6)
- Practice problems

Concepts: 2-sided z-transforms

1. Definition of z-transform
   - For a given DT signal x(n), X(z):= $\sum_{n=-\infty}^{\infty} x(n)z^{-n}$, where z is complex valued.
   - If $x(n)r^{-n}$ is absolutely summable, then X(z) has a finite value where |z|= r.

2. ROC (Region of Convergence)
   - The set of values of z for which the sequence $x(n)z^{-n}$ is absolutely summable, i.e. $\{z \in C : \sum_{n=-\infty}^{\infty} |x(n)z^{-n}| < \infty\}$, where C is the set of complex numbers.
   - Simply put, ROC indicates the region of z where X(z) is finite.
   - By definition, ROC cannot contain any poles.

3. The shape of ROCs:
   - The ROC of an anti-causal signal is of the form |z| <|a|.
   - The ROC of a causal signal is of the form |z| >|a|.
   - The ROC of a two sided signal is of the form |a|<|z| <|b|.
   - The ROC of a finite length signal is the entire z−space except for z=0 and/or z= ∞.

4. Useful z-transformation pairs:
   - If $x(n)= a^n u(n)$, then $X(z) = \frac{z}{z-a}$, ROC = |z| >|a|.
   - If $x(n)= -a^n u(-n-1)$, then $X(z) = \frac{z}{z-a}$, ROC = |z| <|a|.

5. Properties of z-transform: We have x(n)↔ X(z) and $ROC_X= r_2<|z| <r_1$
   - Linearity: $a_1x_1(n)+ a_2x_2(n) \leftrightarrow a_1X_1(z)+ a_2X_2(z)$, ROC≥ $ROC_{X_1} \cap ROC_{X_2}$
   - Time shifting: $x(n-k) \leftrightarrow z^{-k}X(z)$, ROC= ROC_X except z=0 or z= ∞.
   - Scaling in the z-domain: $a^n x(n) \leftrightarrow X(a^{-1}z)$, ROC= $|a|r_2<|z|<|a|r_1$
   - Time reversal: $x(-n) \leftrightarrow X(z^{-1})$, ROC= $r_1<|z|<r_2$
   - Differentiation in the z-domain: $nx(n) \leftrightarrow -z\frac{dX(z)}{dz}$, ROC= ROC_X
   - Convolution: $x_1(n)*x_2(n) \leftrightarrow X_1(z)X_2(z)$, ROC≥$ROC_{X_1} \cap ROC_{X_2}$
   - Correlation: $x_1(n)*x_2(-n) \leftrightarrow X_1(z)X_2(z^{-1})$, ROC≥$ROC_{X_1}(z) \cap ROC_{X_2(z^{-1})}$

Concepts: Inverse z-transform (Partial Fraction Expansions)

1. $X(z) = \frac{b_0 + b_1 z + \ldots + b_m z^m}{a_0 + a_1 z + \ldots + a_N z^N}$ and M≤N

- $\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-p_1} + \ldots + \frac{A_N}{z-p_N}$, where $p_1, p_2, \ldots, p_N$ are roots of $a_0 + a_1 z + \ldots + a_N z^N$.
- Residues: $A_n=(z-p_n)X(z)/z$
- The case of complex poles: $Ap^n+A^*(p^*)^n = 2|A||p|\cos(\omega_0 n+\theta)$, where $A=|A|e^{j\omega 0}$, $p=|p|e^{j\theta}$.

2. $X(z) = \frac{b_0 + b_1 z + \ldots + b_m z^m}{a_0 + a_1 z + \ldots + a_N z^N} = \frac{B(z)}{A(z)}$ and M>N
   - Divide B(z)by A(z) to express it as X(z)= Q(z)+$\frac{R(z)}{A(z)}$ where the degree of R(z) is less than the degree of A(z)
   - Now apply the appropriate partial fraction expansion to $\frac{R(z)}{A(z)}$

Problems

1. Compute the z-transform and the associated ROC's of the following signals

   (a) x(n)=$(\frac{1}{5})^n$ u(n)

   (b) x(n)=2^nu(-n)+$(\frac{1}{3})^n$ u(n)

   (c) x(n)={-1,<u>0</u>,1}

2. Express the z-transform of $y(n) = \sum_{k=-\infty}^{n} x(k)$ in terms of X(z).

3. Using appropriate properties of the z-transforms, determine x(n) for the following transformation

$$X(z)= \log(1 - 0.5z^{-1}),\ |z| > 0.5$$

   Hint: Differentiate X(z)

4. Determine the causal signal x(n)if its z-transform is

$$X(z) = \frac{2 - 1.5z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Recitation 4 Solution - Jung Hyun Bae.

1\. (a) $a^n u(n) \rightarrow \dfrac{z}{z-a}$.

$$\left(\tfrac{1}{5}\right)^n u(n) \rightarrow \frac{z}{z-\frac{1}{5}}.$$

(b) $-a^n u(-n-1) \rightarrow \dfrac{z}{z-a}$

$$2^n u(-n) = 2 \cdot 2^{n-1} u(-(n-1)-1).$$

$$\therefore\ 2^n u(-n) = -\frac{2z}{z-2} \cdot z^{-1} = -\frac{2}{z-2}.$$

$$\left(\tfrac{1}{3}\right)^n u(n) \rightarrow \frac{z}{z-\frac{1}{3}}$$

$$\therefore\ X(z) = \frac{z}{z-\frac{1}{3}} - \frac{2}{z-2}.$$

(c) $X(z) = -1 \cdot z + 0 \cdot 1 + 1 \cdot z^{-1} = -z + z^{-1}$.

2\. $y(n) = x(n) + x(n-1) + \cdots$

$$Y(z) = X(z) + z^{-1}X(z) + \cdots$$

$$= \frac{X(z)}{1-z^{-1}} \quad \text{if} \quad |z| > 1.$$

$$\therefore\ Y(z) = \frac{X(z)}{1-z^{-1}}, \quad \text{ROC}: \ \text{ROC}_x \cap \{|z|>1\}$$

3\. $$-z \cdot \frac{dX(z)}{dz} = \frac{0.5z^{-2}}{1-0.5z^{-1}} \cdot -z = \frac{-0.5z^{-1}}{1-0.5z^{-1}} = 1 - \frac{1}{1-0.5z^{-1}} \leftrightarrow \underbrace{\delta(n) - (0.5)^n u(n)}_{= \, n x(n)}$$

$$\therefore\ x(n) = \frac{1}{n}\delta(n) - \frac{(0.5)^n}{n} u(n).$$

4. $$X(z) = \frac{2z^2 - 1.5z}{z^2 - 1.5z + 0.5}$$

$$\frac{X(z)}{z} = \frac{2z - 1.5}{z^2 - 1.5z + 0.5} = \frac{A_1}{z - \frac{1}{2}} + \frac{A_2}{z - 1}$$

$$A_1 = \frac{X(z)}{z}\left(z - \frac{1}{2}\right)\Big|_{z=\frac{1}{2}} = \frac{2z - 1.5}{z(z-1)}\Big|_{z=\frac{1}{2}} = 2$$

$$A_2 = \frac{X(z)}{z}(z-1)\Big|_{z=1} = \frac{2z - 1.5}{z(z - \frac{1}{2})}\Big|_{z=1} = 1$$

$$\therefore X(z) = \frac{2z}{z - \frac{1}{2}} + \frac{z}{z-1}.$$

i) ROC : $|z| < \frac{1}{2}$.

$$x(n) = -2\left(\tfrac{1}{2}\right)^n u(-n-1) - u(-n-1).$$

ii) ROC : $\frac{1}{2} < |z| < 1$

$$x(n) = 2\cdot\left(\tfrac{1}{2}\right)^n u(n) - u(-n-1)$$

iii) ROC : $|z| > 1$.

$$x(n) = 2\cdot\left(\tfrac{1}{2}\right)^n u(n) + u(n).$$