Recitation 4 - EECS 451, Winter 2010

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OUTLINE

• Review of important concepts (Lecture 5-6)
• Practice problems

Concepts: 2-sided z-transforms

1. Definition of z-transform

   • For a given DT signal \( x(n) \), \( X(z) := \sum_{n=-\infty}^{\infty} x(n)z^{-n} \), where \( z \) is complex valued.
   
   • If \( x(n)r^{-n} \) is absolutely summable, then \( X(z) \) has a finite value where \( |z|= r \).

2. ROC (Region of Convergence)

   • The set of values of \( z \) for which the sequence \( x(n)z^{-n} \) is absolutely summable, i.e.
     \[ \{ z \in \mathbb{C} : \sum_{n=-\infty}^{\infty} |x(n)z^{-n}| < \infty \} \]
     where \( C \) is the set of complex numbers.
   
   • Simply put, ROC indicates the region of \( z \) where \( X(z) \) is finite.
   
   • By definition, ROC cannot contain any poles.

3. The shape of ROCs:

   • The ROC of an anti-causal signal is of the form \( |z| <|a| \).
   
   • The ROC of a causal signal is of the form \( |z| >|a| \).
   
   • The ROC of a two sided signal is of the form \( |a|<|z| <|b| \).
   
   • The ROC of a finite length signal is the entire \( z \)-space except for \( z=0 \) and/or \( z=\infty \).

4. Useful z-transformation pairs:

   • If \( x(n) = a^n u(n) \), then \( X(z) = \frac{z}{z-a} \), ROC = \( |z| >|a| \).
   
   • If \( x(n) = -a^{-n}(n-1) \), then \( X(z) = \frac{z}{z-a} \), ROC = \( |z| <|a| \).

5. Properties of z-transform: We have \( x(n) \leftrightarrow X(z) \) and \( R\text{OC}_{x} = r_{2}<|z| <r_{1} \)

   • Linearity: \( a_{1}x_{1}(n)+a_{2}x_{2}(n) \leftrightarrow a_{1}X_{1}(z)+a_{2}X_{2}(z) \), ROC \( \geq R\text{OC}_{x_{1}} \cap R\text{OC}_{x_{2}} \)

   • Time shifting: \( x(n-k) \leftrightarrow z^{-k}X(z) \), ROC = \( R\text{OC}_{x} \) except \( z=0 \) or \( z=\infty \).

   • Scaling in the \( z \)-domain: \( a^{n}x(n) \leftrightarrow X(a^{-1}z) \), ROC = \( |a| r_{2}<|z| <|a|r_{1} \)

   • Time reversal: \( x(-n) \leftrightarrow X(z^{-1}) \), ROC = \( r_{1}<|z| <r_{2} \)

   • Differentiation in the \( z \)-domain: \( nx(n) \leftrightarrow -\frac{dX(z)}{dz} \), ROC = \( R\text{OC}_{x} \)

   • Convolution: \( x_{1}(n)\ast x_{2}(n) \leftrightarrow X_{1}(z)X_{2}(z) \), ROC \( \geq R\text{OC}_{x_{1}} \cap R\text{OC}_{x_{2}} \)

   • Correlation: \( x_{1}(n)\ast x_{2}(-n) \leftrightarrow X_{1}(z)X_{2}(z^{-1}) \), ROC \( \geq R\text{OC}_{x_{1}}(z) \cap R\text{OC}_{x_{2}} \)

Concepts: Inverse z-transform (Partial Fraction Expansions)

1. \( X(z) = \frac{b_{0}+b_{1}z+\ldots+b_{M}z^{n}}{a_{0}+a_{1}z+\ldots+a_{N}z^{N}} \) and \( M \leq N \)
\[ X(z) = \frac{A_0}{z} + \frac{A_1}{z-p_1} + \ldots + \frac{A_N}{z-p_N} \]

where \( p_1, p_2, \ldots, p_N \) are roots of \( a_0 + a_1 z + \ldots + a_N z^N \).

- Residues: \( A_n = (z-p_n)X(z)/z \)

- The case of complex poles: \( A^*p = \frac{|A||p|\cos(\omega_0 n + \theta)}{2} \), where \( A = |A|e^{j\omega_0}, p = |p|e^{j\theta} \).

2. \( X(z) = \frac{b_0 + b_1 z^2 + \ldots + b_n z^n}{a_0 + a_1 z + \ldots + a_n z^m} = \frac{B(z)}{A(z)} \) and \( M > N \)

- Divide \( B(z) \) by \( A(z) \) to express it as \( X(z) = Q(z) + \frac{R(z)}{A(z)} \) where the degree of \( R(z) \) is less than the degree of \( A(z) \)

- Now apply the appropriate partial fraction expansion to \( \frac{R(z)}{A(z)} \)

Problems

1. Compute the z-transform and the associated ROC’s of the following signals
   (a) \( x(n) = (\frac{1}{5})^n u(n) \)
   (b) \( x(n) = 2^n u(-n) + (\frac{1}{3})^n u(n) \)
   (c) \( x(n) = \{-1, 0, 1\} \)

2. Express the z-transform of \( y(n) = \sum_{k=-\infty}^{\infty} y(k) \) in terms of \( X(z) \).

3. Using appropriate properties of the z-transforms, determine \( x(n) \) for the following transformation
   \[ X(z) = \log(1 - 0.5z^{-1}), |z| > 0.5 \]
   Hint: Differentiate \( X(z) \)

4. Determine the causal signal \( x(n) \) if its z-transform is
   \[ X(z) = \frac{2 - 1.5z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}} \]
1. (a) \[ a^n u(n) \to \frac{z}{z-a} \]

\[ (\frac{1}{2})^n u(n) \to \frac{z}{z-\frac{1}{2}} \]

(b) \[-a^n u(-n-1) \to \frac{z}{z-a} \]

\[ 2^n u(-n) = 2 \cdot 2^{n-1} u(-(n-1)-1) \]

\[ = \frac{2z}{z-2}, \quad z^{-1} = \frac{2}{z-2} \]

(c) \[ x(z) = \frac{z}{z^{-\frac{1}{2}}} - \frac{2}{z-2} \]

\[ x(z) = -1, z + 0.1 + 1, z^{-1} = -z + z^{-1} \]

2. \[ y(n) = x(n) + x(n-1) + \cdots \]

\[ y(z) = x(z) + z^{-1} x(z) + \cdots \]

\[ = \frac{x(z)}{1-z^{-1}} \quad \text{if} \quad |z|1. \]

\[ y(z) = \frac{x(z)}{1-z^{-1}}, \quad \text{ROC} : \quad \text{ROC}_x \cap \{ |z| > 1 \} \]

3. \[-z \cdot \frac{d}{dz} x(z) = \frac{0.5 z^{-2}}{1-0.5 z^{-1}} \cdot z = \frac{-0.5 z^{-1}}{1-0.5 z^{-1}} = 1 - \frac{1}{1-0.5 z^{-1}} \quad \leftrightarrow \quad d(n) = (0.5)^n u(n) \]

\[ x(n) = \frac{1}{n} d(n) - \frac{(0.5)^n}{n} u(n) \]

\[ n x(n) \]

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4. $X(z) = \frac{2z^2 - 1.5z}{z^2 - 1.5z + 0.5}$

\[ \frac{X(z)}{z} = \frac{2z - 1.5}{z^2 - 1.5z + 0.5} = \frac{A_1}{z - \frac{1}{2}} + \frac{A_2}{z - 1} \]

\[ A_1 = \left. \frac{X(z)}{z} \right|_{z = \frac{1}{2}} = \left. \frac{2z - 1.5}{z(z - 1)} \right|_{z = \frac{1}{2}} = 2 \]

\[ A_2 = \left. \frac{X(z)}{z} \right|_{z = 1} = \left. \frac{2z - 1.5}{z(z - \frac{1}{2})} \right|_{z = 1} = 1 \]

\[ \therefore X(z) = \frac{2z}{z - \frac{1}{2}} + \frac{z}{z - 1}. \]

i) ROC: \( |z| < \frac{1}{2} \)

\[ X(n) = -2 \left( \frac{1}{2} \right)^n u(-n-1) - u(-n-1) \]

ii) ROC: \( \frac{1}{2} < |z| < 1 \)

\[ X(n) = 2 \left( \frac{1}{2} \right)^n u(n) - u(-n-1) \]

iii) ROC: \( |z| > 1 \)

\[ X(n) = 2 \left( \frac{1}{2} \right)^n u(n) + u(n) \]