Recitation 1 -EECS 451, Winter 2010

Jan 13, 2010

OUTLINE

- Review of Fourier Transform
- Practice problems

Concepts

1. Fourier Transforms
   - Definition
   - Basic Properties
   - Conjugate symmetry

Problems

1. Determine the Fourier transform of the signal $x(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$.

   (a) Do you observe anything peculiar about the Fourier transform of this signal?

   (b) Which other signal exhibits a similar property?

2. Compute the Fourier transform of each of the following signals

   (a) $x(t)= e^{-|t|}\cos 2t$

   (b) The signal x(t) depicted in figure 1

3. Fig 2 shows an incomplete Fourier transform of a sinusoidal signal x(t). Determine the signal x(t) using the information given below.

   (a) x(t) is a real signal comprising of two nonzero frequencies.

   (b) The value of the signal at time t =0 is 7.

   (c) The energy of the signal is $(1+54\pi)/(2\pi)$

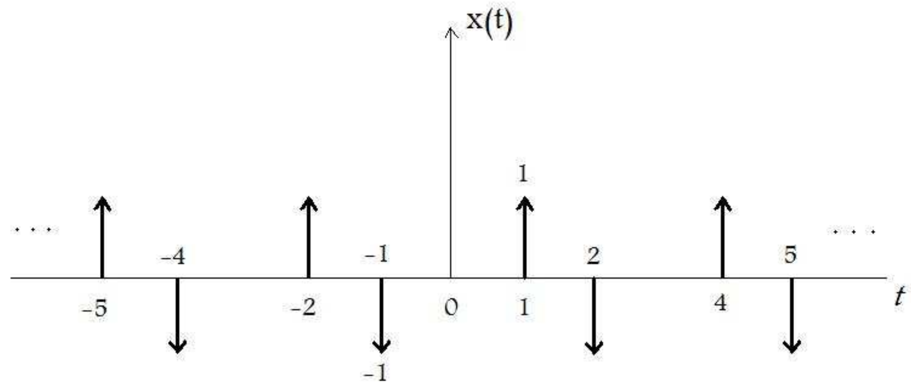

Figure 1:

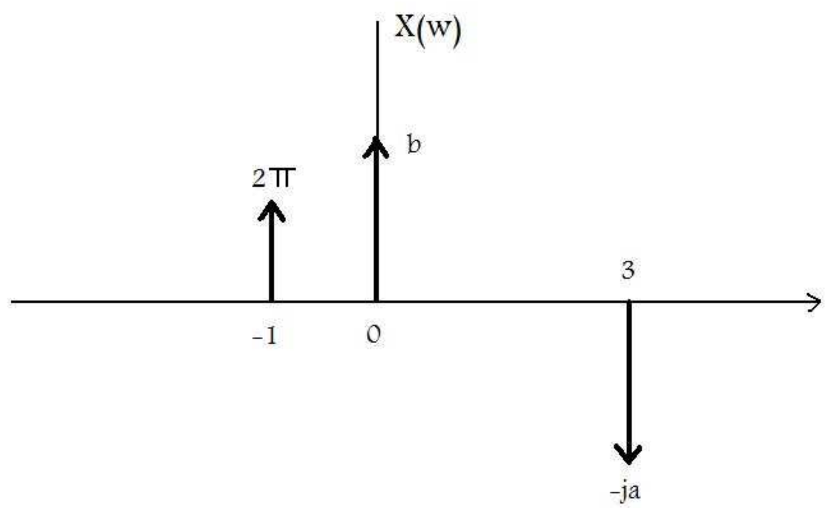

Figure 2:

1. $X(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t+j\omega)^2} e^{-\frac{1}{2}\omega^2} dt$$

(Let $t' = t + j\omega$)

$$= e^{-\frac{1}{2}\omega^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t'^2}{2}} dt'$$

Since $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t'^2}$ is a pdf of a Gaussian R.V.,

$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t'^2}{2}} dt' = 1$.

$\therefore X(\omega) = e^{-\frac{1}{2}\omega^2}$.

We can also prove $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 1$ as follows.

Since $\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ is non-negative for all t, it suffices to show that

$$\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt\right]\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du\right] = 1.$$

$$\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt\right]\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{t^2+u^2}{2}} dt\, du$$

(Let $t = r\cos\theta$, $u = r\sin\theta$, i.e. change the cordinate)

$$= \int_{0}^{2\pi}\int_{0}^{\infty} \frac{1}{2\pi} r e^{-\frac{r^2}{2}} dr\, d\theta$$

$$= \int_{0}^{2\pi} \frac{1}{2\pi} \left[-e^{-\frac{r^2}{2}}\right]_0^{\infty} d\theta$$

$$= \int_{0}^{2\pi} \frac{1}{2\pi} d\theta = 1.$$

(a) The original signal and the transform of it have the same form. In other words, $x(t)$ is an eigen function of FT.

(b) Eigen functions of FT are called 'Hermite functions'. One of them is $te^{-\frac{t^2}{2}}$.

2. (a) $x(t) = e^{-|t|} \cos 2t$

$$= e^{-|t|} \frac{e^{j2t} + e^{-j2t}}{2}.$$

Let $Y(w)$ be the FT of $e^{-|t|}$.

Then, $X(w) = \frac{1}{2}(Y(w-2) + Y(w+2))$ ($\because$ modulation property)

$$Y(w) = \int_{-\infty}^{\infty} e^{-|t|} e^{-jwt} dt$$

$$= \int_{0}^{\infty} e^{-t} e^{-jwt} dt + \int_{-\infty}^{0} e^{t} e^{-jwt} dt$$

$$= \frac{1}{1+jw} + \frac{1}{1-jw} = \frac{2}{1+w^2}$$

$$\therefore X(w) = \frac{1}{1+(w-2)^2} + \frac{1}{1+(w+2)^2}.$$

(b) Let $x_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-3k)$.

Then, $x(t) = x_1(t-1) - x_1(t-2)$.

Hence, $X(w) = e^{-jw} X_1(w) - e^{-j2w} X_1(w)$

$$= (e^{-jw} - e^{-j2w}) X_1(w).$$

Consider now the Fourier series representation of $x_1(t)$.

Then. $x_1(t) = \frac{1}{3} \sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{3}kt}$.

Since $1 \xrightarrow{\mathcal{F}} 2\pi \delta(w)$ and by the modulation property,

$$X_1(w) = \frac{2\pi}{3} \sum_{k=-\infty}^{\infty} \delta(w - \frac{2\pi}{3}k).$$

$$\therefore X(w) = \frac{2\pi}{3}(e^{-jw} - e^{-j2w}) \sum_{k=-\infty}^{\infty} \delta(w - \frac{2\pi}{3}k).$$

★ Please change (c) to 'The energy of the signal is $\frac{1+54\pi}{2\pi}$'.

3. Since $x(t)$ has two nonzero frequencies, we know that there is no other frequency components than $0, \pm 1, \pm 3$.

Since $x(t)$ is real, $X(-\omega) = X^*(\omega)$ by conjugate symmetry.

Therefore, $X(-3) = ja$, $X(1) = 2\pi$.

From (b), $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega = 7$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega = \frac{1}{2\pi}(4\pi + b)$$

$\therefore\ b = 10\pi$.

Fourier series coefficient of $x(t)$.

From (c), $\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt \overset{(i)}{=} \sum_k |a_k|^2 = \frac{1+54\pi}{2\pi} = \frac{1}{2\pi} + 27$

where (i) is from Parseval's theorem for periodic signal, and T_0 is the period of $x(t)$. Note that $x(t)$ is a periodic function. We can see that from the fact that the FT of $x(t)$ is discrete.

Note that $X(\omega) = ja\delta(t+2) + 2\pi\delta(t+1) + 10\pi\delta(t) + 2\pi\delta(t-1) - ja\delta(t-2)$

Then, by the definitions of FT and FS,

a_k's are $\frac{ja}{2\pi}, 1, 5, 1, -\frac{ja}{2\pi}$.

Therefore. $\sum_k |a_k|^2 = \frac{a^2}{2\pi^2} + 27 = \frac{1}{2\pi} + 27$.

$\therefore\ a = \pm\sqrt{\pi}$.

Then, $x(t) = 5 + 2\cos t \pm \frac{1}{\sqrt{\pi}} \sin 2t$.

We can prove that

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \sum_k |a_k|^2 \quad \text{as follows.}$$

$$x(t) = \sum_k a_k e^{j\frac{2\pi}{T_0}kt}, \quad \text{where} \quad a_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-j\frac{2\pi}{T_0}kt}\, dt.$$

Therefore, $\frac{1}{T_0}\int_{T_0} x^*(t)\, e^{-j\frac{2\pi}{T_0}kt}\, dt$

$= a_{-k}^*$,

and hence, $x^*(t) = \sum_k a_{-k}^* e^{j\frac{2\pi}{T_0}kt}$.

Let $x(t)x^*(t) = \sum_k c_k e^{j\frac{2\pi}{T_0}kt}$.

Then, $c_k = \frac{1}{T_0}\int_{T_0} x(t)\, x^*(t)\, e^{-j\frac{2\pi}{T_0}kt}\, dt$

$$= \frac{1}{T_0}\int_{T_0} \left[\sum_l a_l e^{j\frac{2\pi}{T_0}lt}\right]\left[\sum_m a_{-m}^* e^{j\frac{2\pi}{T_0}mt}\right] e^{-j\frac{2\pi}{T_0}kt}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} \sum_l \sum_m a_l a_{-m}^*\, e^{j\frac{2\pi}{T_0}t(l+m-k)}\, dt.$$

$$= \sum_l \sum_m a_l a_{-m}^* \int_{T_0} \frac{1}{T_0} e^{j\frac{2\pi}{T_0}t(l+m-k)}\, dt.$$

$$\left(\text{Since} \quad \frac{1}{T_0}\int_{T_0} e^{j\frac{2\pi}{T_0}kt}\, dt = \begin{cases} 1, & k=0 \\ 0, & \text{othewise} \end{cases}\right).$$

$$= \sum_l a_l a_{l-k}^*,$$

Then, $\frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt$

$$= \frac{1}{T_0}\int_{T_0} x(t)\, x^*(t)\, dt$$

$$= \frac{1}{T_0}\int_{T_0} \sum_k c_k e^{j\frac{2\pi}{T_0}kt}\, dt$$

$$= c_0$$

$$= \sum_l a_l a_l^* = \sum_l |a_l|^2.$$

We can prove the similar thing for non periodic signal as follows.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(w)\, e^{-jwt} \right] dw\, dt \qquad \left(\because\ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dw \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(w) \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt\, dw \qquad (\because \text{ change the order of integration})$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(w)\, X(w)\, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(w)|^2\, dw.$$