TODAY'S LECTURE

1. 2-sided z-transform:
   Incredibly useful tool.
2. 1-sided z-transform:
   Includes initial conditions.
   Next time: Inverse z-xform.

2-SIDED z-TRANSFORM

DEF: $\mathcal{Z}\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

Compare to: Laplace transform:
$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$.

ROC: $\{z : \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converges$\}$
ROC: Always is an annulus or ring:
ROC: $r_1 < |z| < r_2$ for some r_1, r_2.

FINITE-LENGTH SIGNALS [1/2]

Then: X(z)=polynomial in z^{-1} with coefficients $x[n]$.

EX: $x[n] = \{\ldots 0, 0, 3, 1, \underline{4}, 2, 5, 0, 0 \ldots\}$
Note: $\underline{4}$ means x[0]=4; sets $n = 0$.

Then: $X(z)=3z^2+z+4+2z^{-1}+5z^{-2}=(3z^4+z^3+4z^2+2z+5)/z^2$.
ROC: $0 < |z| < \infty$ (converges except at 0 and ∞).

FINITE-LENGTH SIGNALS [2/2]

1. $\mathcal{Z}\{\delta[n]\} = 1$.
2. $\mathcal{Z}\{\delta[n-3]\} = z^{-3}$.
3. $\mathcal{Z}\{3\delta[n+2]\} = 3z^2$.

Can: Read off inverse z-transform.

CAUSAL GEOMETRIC SIGNALS

Fact $\mathcal{Z}\{a^n u[n]\}=\frac{z}{z-a}$. **ROC:** $\{z : |z| > |a|\}$.

Proof: $\mathcal{Z}\{a^n u[n]\}=\sum_{n=0}^{\infty} a^n z^{-n}=\frac{1}{1-az^{-1}}=\frac{z}{z-a}$.
ROC: $|az^{-1}| < 1 \rightarrow |z| > |a|$ for series to converge.

EX: $\mathcal{Z}\{(-\frac{1}{3})^n u[n]\}=\frac{1}{1+\frac{1}{3}z^{-1}}=\frac{z}{z+\frac{1}{3}}$. **ROC:** $|z| > |-\frac{1}{3}| = \frac{1}{3}$.

ANTICAUSAL GEOMETRIC SIGNALS

Fact: $\mathcal{Z}\{-a^n u[-n-1]\} = \frac{z}{z-a}$. **ROC:** $\{z : |z| < |a|\}$.

Proof: $\mathcal{Z}\{-a^n u[-n-1]\}=-\sum_{n=-\infty}^{-1}(\frac{a}{z})^n$
$= -\sum_{n=1}^{\infty}(\frac{z}{a})^n=\frac{-a^{-1}z}{1-a^{-1}z}=\frac{1}{1-az^{-1}}=\frac{z}{z-a}$.
ROC: $|a^{-1}z| < 1 \rightarrow |z| < |a|$ for series to converge.

EX: $\mathcal{Z}\{-(2)^n u[-n-1]\}=\frac{1}{1-2z^{-1}}=\frac{z}{z-2}$. **ROC:** $|z| < 2$.

GEOMETRIC SIGNALS

Huh? 2 different signals have same z-transform: $\mathcal{Z}\{a^n u[n]\}=\mathcal{Z}\{-a^n u[-n-1]\} = \frac{z}{z-a}$.

ROC: The ROCs of the two signals differ:
ROC: $\{z : |z| > |a|\}$ and $\{z : |z| < |a|\}$.
So: ROC DOES matter quite a bit!

TWO-SIDED SIGNALS

Fact: $\mathcal{Z}\{a^n u[n] + b^n u[-n-1]\} = \frac{z}{z-a} - \frac{z}{z-b}$.

ROC: $|a| < |z| < |b|$ (annulus or ring). Watch signs!

Note: If $|b| < |a|$ then ROC=empty (*never* converges)!
Note: z-xform of sum→ROC=$\bigcap$(ROCs of each term).

TWO-SIDED SIGNALS: EX. #1

Goal: Compute $\mathcal{Z}\{(\frac{1}{2})^n u[n]+(2)^n u[-n-1]\}$.
Soln: $\frac{z}{z-\frac{1}{2}}\frac{z-2}{z-2}-\frac{z}{z-2}\frac{z-\frac{1}{2}}{z-\frac{1}{2}}=\frac{-1.5z}{z^2-2.5z+1}$.
ROC: $\{z : \frac{1}{2} < |z| < 2\}$. Annulus or ring.

CAUSAL SINUSOIDAL SIGNALS

Fact: $\mathcal{Z}\{\cos(\omega_0 n)u[n]\} = \frac{z^2-z\cos(\omega_0)}{z^2-2z\cos(\omega_0)+1}$. **ROC:** $\{z : |z| > 1\}$.

Proof $\mathcal{Z}\{\cos(\omega_0 n)u[n]\} = \frac{1}{2}\mathcal{Z}\{e^{j\omega_0 n}u[n]\}+\frac{1}{2}\mathcal{Z}\{e^{-j\omega_0 n}u[n]\}$

$= \frac{1}{2}\frac{z}{z-e^{j\omega_0}}\frac{z-e^{-j\omega_0}}{z-e^{-j\omega_0}}+\frac{1}{2}\frac{z}{z-e^{-j\omega_0}}\frac{z-e^{j\omega_0}}{z-e^{j\omega_0}} = \frac{z^2-z\cos(\omega_0)}{z^2-2z\cos(\omega_0)+1}$.

Note: Over common denominator. **ROC:** $|z| > |e^{\pm j\omega_0}| = 1$.

TWO-SIDED SIGNALS: POLES

Fact: $\mathcal{Z}\{\sum_i A_i p_i^n u[n]+\sum_j B_j q_j^n u[-n-1]\} = \sum_i \frac{A_i z}{z-p_i}-\sum_j \frac{B_j z}{z-q_j}$.

ROC: $|\underset{x_{\text{causal}}[n]}{\text{largest pole of}}| < |z| < |\underset{x_{\text{anticausal}}[n]}{\text{smallest pole of}}|$.

Small (magnitude) poles↔causal part;
Large (magnitude) poles↔anticausal part.
ROC is always an annulus (ring)
whose radii are *successive* poles.

TWO-SIDED SIGNALS: EX. #2

Goal: Compute $\mathcal{Z}\{(2)^n u[n] + (\frac{1}{2})^n u[-n-1]$.
Soln: Does not exist! Why not?
ROC: $\{z : |z| > 2 \cap |z| < \frac{1}{2}\} = \emptyset$.
That is: The series never converges!

TWO-SIDED SIGNALS: EX. #3

Goal: Compute $\mathcal{Z}\{9(2)^n u[n]+8(3)^n u[n]$
$-7(4)^n u[-n-1]+6(5)^n u[-n-1]\}$.

Soln: $\frac{9z}{z-2}+\frac{8z}{z-3}+\frac{7z}{z-4}-\frac{6z}{z-5}$. Watch signs!

ROC: $\{z : 3 < |z| < 4\}$ since this equals
$\{|z| > 2\} \cap \{|z| > 3\} \cap \{|z| < 4\} \cap \{|z| < 5\}$

PROPERTIES OF z-TRANSFORMS [1/3]

1. $\mathcal{Z}\{x[n] * y[n]\} = X(z)Y(z)$.

That is: z-transforms convert discrete convolution to multiplication!

Compare: $\mathcal{L}\{x(t) * y(t)\} = X(s)Y(s)$.
$* \Leftrightarrow$polynomial multiplication.

PROPERTIES OF z-TRANSFORMS [3/3]

2. z^n are eigenfunctions of LTI systems

Huh? $z_o^n \rightarrow \boxed{\mathbf{h[n]}} \rightarrow z_o^n H(z_o)$

Compare: $e^{s_o t} \rightarrow \boxed{\mathbf{h(t)}} \rightarrow e^{s_o t} H(s_o)$.

Proof: $y[n]=h[n]*z_o^n=\sum h[i] z_o^{n-i} = z_o^n \sum h[i] z_o^{-i} = z_o^n H(z_o)$. QED.

So? Set $z_o = e^{j\omega_o}$ for *frequency response*.

MISCELLANEOUS

Goal: Compute $\mathcal{Z}\{\frac{1}{n} u[n-1]\}$.

Soln: $\mathcal{Z}\{\frac{1}{n} u[n-1]\}=\sum_{n=1}^{\infty} \frac{z^{-n}}{n} = -\log(1-z^{-1})$.

ROC: $\{z : |z| > 1\}$. Power series expansion.
I'm required to do this; now forget it.

PROPERTIES OF z-TRANSFORMS [2/3]

This has enormous implications.
Useful as $\mathcal{L}$ is in cont. time.

EX: Convolution is associative:
$(x[n] * y[n]) * z[n] = x[n] * (y[n] * z[n])$

Proof: [X(z)Y(z)]Z(z)=X(z)[Y(z)Z(z)]. QED.

EX: Convolution is distributive.

Proof: X(z)[Y(z)+Z(z)]=X(z)Y(z)+X(z)Z(z). QED.

1-SIDED z-TRANSFORM

DEF: $\mathcal{Z}^+\{x[n]\} = X^+(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$.

But: Use this **even if** $x[n]$ **is noncausal**.

Compare: 1-sided Laplace transform
$\mathcal{L}\{x(t)\} = \int_0^{\infty} x(t) e^{-st} dt$.

Why? Use to solve difference equations with nonzero initial conditions.

PROPERTIES OF 1-SIDED $\mathcal{Z}$ [1/2]

1a. $D>0 \rightarrow \mathcal{Z}^+\{x[n-D]\} = z^{-D}(X^+(z) + \sum_{n=1}^{D} x[-n]z^n)$.

1b. $D>0 \rightarrow \mathcal{Z}^+\{x[n+D]\} = z^{+D}(X^+(z) - \sum_{n=0}^{D-1} x[n]z^{-n})$.

- i. Shift right→must add in terms *previously* anticausal.
- ii. Shift left→must subtract off terms *now* anticausal.
- iii. Compare to $\mathcal{L}\{\frac{d^2x}{dt^2}\} = s^2X(s) - sx(0) - \frac{dx}{dt}(0)$.

PROPERTIES OF 1-SIDED $\mathcal{Z}$ [2/2]

Why? Consider this example (shows what's happening):

Given: $x[n] = \{3, \underline{1}, 4\} \rightarrow X^+(z) = 1 + 4z^{-1}$.

Delay: $y[n] = \{\underline{3}, 1, 4\} \rightarrow Y^+(z) = 3 + z^{-1} + 4z^{-2}$.

Note: $Y^+(z) = z^{-1}X^+(z) + 3 = z^{-1}(X^+(z) + 3z)$. 3=x[–1].

2. Other properties same as 2-sided $\mathcal{Z}$.

ROC: Always $|z| > |p_{max}|$ (largest pole of $X^+(z)$).

So: Don't bother with ROC; $x[n]$ always causal.

SOLVING DIFFERENCE EQUATIONS WITH INITIAL CONDITIONS [1/4]

Solve: 2y[n]+3y[n–1]+y[n–2]=u[n]+u[n–1]–u[n–2]

with: Initial conditions y[–1]=2 and y[–2]=–1.

using: 1-sided z-transform (Chen p.264)

Warning! Algebra alert next 3 slides!

EXAMPLE, CONT. [2/4]

$\mathcal{Z}^+$ of: 2y[n]+3y[n–1]+y[n–2]=u[n]+u[n–1]–u[n–2]

Get: $2Y^+(z) + 3z^{-1}[Y^+(z) + y[-1]z] + z^{-2}[Y^+(z) + y[-1]z + y[-2]z^2]$

$= (1 + z^{-1} - z^{-2})U(z) = \text{rhs}$.

Collect: $[2 + 3z^{-1} + z^{-2}]Y^+(z) + [(3y[-1] + y[-2]) + z^{-1}y[-1]] = \text{rhs}$.

Solve:

$$Y^+(z) = \underbrace{\frac{z^2+z-1}{2z^2+3z+1}U(z)}_{\text{zero-state response}} - \underbrace{\frac{(3y[-1]+y[-2])z^2 + y[-1]z}{2z^2+3z+1}}_{\text{zero-input response}}$$

EXAMPLE, CONT. [3/4]

Plug in: $u[n]$=unit step→ $U(z) = U^+(z) = \frac{z}{z-1}$.

and: *Initial conditions* y[–1]=2 & y[–2]=–1.

$$Y^+(z) = \frac{z^3+z^2-z}{(2z^2+3z+1)(z-1)} - \frac{5z^2+2z}{2z^2+3z+1}\frac{z-1}{z-1}.$$

$$Y^+(z) = \frac{-4z^3+4z^2+z}{2(z+\frac{1}{2})(z+1)(z-1)}$$ COMMON DENOMINATOR

EXAMPLE, CONT. [4/4]

Partial fraction: $Y^+(z) = \frac{4}{3}\frac{z}{z+\frac{1}{2}} - \frac{7}{2}\frac{z}{z+1} + \frac{1}{6}\frac{z}{z-1}$.

Matlab: `[R P]=residue([-4 4 1],[conv([2 3 1],[1 -1])])`

$\mathcal{Z}^{-1}$:

$$\text{y}[n] = \underbrace{\frac{4}{3}(-\frac{1}{2})^n u[n] - \frac{7}{2}(-1)^n u[n]}_{\text{natural response (like h[n])}} + \underbrace{\frac{1}{6}u[n]}_{\text{forced}}$$