TOPICS FOR TODAY'S LECTURE

1. Data Windows for Spectra
   Using: DFT, FFT, *, sincs.
2. Comparison of the various
   $\subset \{D, F, S, T\}$ transforms.

BASIC PROBLEM [1/3]

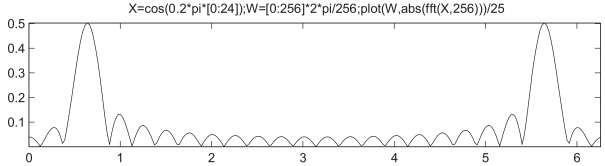
Given: y[n]=$A_1 \cos(\omega_1 n + \theta_1) + A_2 \cos(\omega_2 n + \theta_2)$.
Data: for $\{y[n], 0 \le n \le L-1\}$ (length=L).
Goal: Compute unknown ω_i from this data.

Idea: Compute $Y_k = \sum_{n=0}^{N-1} y[n] e^{-j2\pi nk/N}$.
Then: Find peaks in $|Y_k|$ at k=$k_1, k_2 < \frac{N}{2}$.
Get: The discrete frequencies are: $\omega_i = \frac{2\pi}{N} k_i$.

BASIC PROBLEM [2/3]

EX: $\omega = 0.2\pi$; L=25; N=256.

X=cos(0.2*pi*[0:24]);W=[0:256]*2*pi/256;plot(W,abs(fft(X,256)))/25



BASIC PROBLEM [3/3]

WTF? y[n]=$\begin{cases} \cos(\omega_o n) & 0 \le n \le \text{L}-1 \\ 0 & \text{otherwise} \end{cases}$ Let w[n]=$\begin{cases} 1 & 0 \le n \le \text{L}-1 \\ 0 & \text{otherwise} \end{cases}$

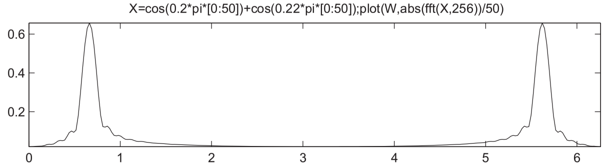
$\Leftrightarrow$ y[n]=$\cos(\omega_o n)$w[n] where w[n]=rectangular window
$\Leftrightarrow$ $Y(e^{j\omega}) = \frac{1}{2} W(e^{j(\omega-\omega_o)}) + \frac{1}{2} W(e^{j(\omega+\omega_o)})$ (modulation)

where: $W(e^{j\omega})$=DTFT$\{w[n]\} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$

RESOLVING PEAKS [1/4]

EX: $\omega = 0.2\pi$ & 0.22π; L=50; N=256.

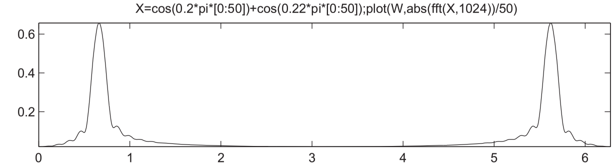
X=cos(0.2*pi*[0:50])+cos(0.22*pi*[0:50]);plot(W,abs(fft(X,256))/50)



RESOLVING PEAKS [2/4]

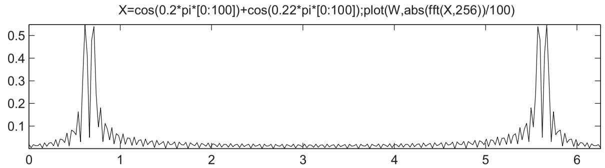
EX: $\omega = 0.2\pi$ & 0.22π; L=50; N=1024.

X=cos(0.2*pi*[0:50])+cos(0.22*pi*[0:50]);plot(W,abs(fft(X,1024))/50)



RESOLVING PEAKS [3/4]

EX: $\omega = 0.2\pi$ & 0.22π; L=100; N=256.



RESOLVING PEAKS [4/4]

1. Increasing N=DFT order no help.
2. Increasing L=data length helps.
3. **Need: roughly** $|\omega_1 - \omega_2| > \frac{2\pi}{L}$ to resolve peaks at ω_1 and ω_2.

Since: The first zero-crossing of $W(e^{j\omega})$ occurs at $\omega = \frac{2\pi}{L} \rightarrow$ peaks overlap.

DATA WINDOWS [1/4]

NO: Not Microsoft Windows!
NO: "I don't do (data) windows."

Idea: Instead of w[n]=1 for $0 \leq n \leq$ L–1
Use: Some other *data window* w[n].

Tradeoff: Main lobe width (gives resolution)
vs. Side lobe amplitudes (ripple).

DATA WINDOWS [2/4]

Table: Commonly used data windows w[n].
Length: w[n]=0 unless $0 \leq n \leq L$ (not L–1).

MAIN: Main lobe width (gives resolution)

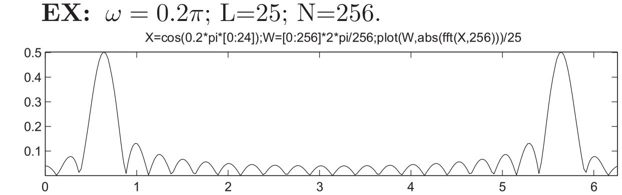
SIDE: Amplitude ratio in dB: $\frac{1^{\text{st}}\,\text{SIDE LOBE}}{\text{MAIN LOBE}}$.

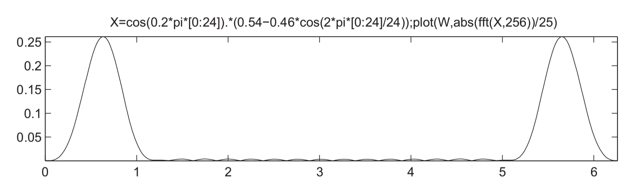
Note: Slight changes make big difference!

DATA WINDOWS [3/4]

NAME	FORMULA	MAIN	SIDE
Rectangle	1	$4\pi/L$	-13
Bartlett	$1-2\lvert n-\frac{L}{2}\rvert/L$	$8\pi/L$	-27
Hanning	$0.50-0.50\cos(\frac{2\pi n}{L})$	$8\pi/L$	-32
Hamming	$0.54-0.46\cos(\frac{2\pi n}{L})$	$8\pi/L$	-43
Blackman	$0.42 {}^{-0.50\cos(\frac{2\pi n}{L})}_{+0.08\cos(\frac{4\pi n}{L})}$	$12\pi/L$	-58

EX: $\omega = 0.2\pi$; L=25; N=256.





CONTINUOUS AND DISCRETE:
FOURIER TRANSFORMS [1/5]

CONTINUOUS TIME	DISCRETE TIME
$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$
$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
$\mathcal{F}\{x(t-T)\} = X(\omega) e^{-j\omega T}$	$\mathrm{dtft}\{x[n-N]\} = X(e^{j\omega}) e^{-j\omega N}$

CONTINUOUS AND DISCRETE:
FOURIER SERIES [2/5]

CONTINUOUS TIME	DISCRETE TIME
$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}$	$\mathrm{DTFS}: x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$
$X_k = \frac{1}{T} \int_0^T x(t) e^{-j2\pi kt/T} dt$	$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}$
$X_k = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}$	$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$

CONTINUOUS AND DISCRETE:
LAPLACE AND Z TRANSFORMS [3/5]

CONTINUOUS TIME	DISCRETE TIME
$\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$	$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$
$\mathcal{L}\{e^{at} u(t)\} = 1/(s-a)$	$\mathcal{Z}\{a^n u[n]\} = z/(z-a)$
$\{s : Re[s] > Re[a]\}$	$\{z : \lvert z\rvert > \lvert a\rvert\}$

CONTINUOUS AND DISCRETE:
ONE-SIDED TRANSFORMS [4/5]

CONTINUOUS TIME	DISCRETE TIME
$\mathcal{L}\{x(t)\} = \int_0^{\infty} x(t) e^{-st} dt$	$\mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}$
$\mathcal{L}\{\frac{dx}{dt}\} = sX(s) - x(0)$	$\mathcal{Z}\{x[n-1]\} = z^{-1} X(z) + x[-1]$

CONTINUOUS AND DISCRETE:
SINC FUNCTIONS [5/5]

CONTINUOUS TIME	DISCRETE TIME
$\mathcal{F}\begin{cases} 1 & \lvert t\rvert < \frac{T}{2} \\ 0 & \lvert t\rvert > \frac{T}{2} \end{cases} = \frac{\sin(\omega T/2)}{\omega/2}$	$\mathrm{dtft}\begin{cases} 1 & \lvert n\rvert \le \frac{N-1}{2} \\ 0 & \lvert n\rvert > \frac{N-1}{2} \end{cases} = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$
$\mathcal{F}^{-1}\begin{cases} 1 & \lvert\omega\rvert < \mathrm{W} \\ 0 & \lvert\omega\rvert > \mathrm{W} \end{cases} = \frac{\sin(Wt)}{\pi t}$	$\mathrm{dtft}^{-1}\begin{cases} 1 & \lvert\omega\rvert < \mathrm{W} \\ 0 & \lvert\omega\rvert > \mathrm{W} \end{cases} = \frac{\sin(Wn)}{\pi n}$
$\mathcal{F}\{x(t)\} = \mathcal{L}\{x(t)\}\rvert_{s=j\omega}$	$\mathrm{dtft}\{x[n]\} = \mathcal{Z}\{x[n]\}\rvert_{z=e^{j\omega}}$