TOPICS FOR TODAY'S LECTURE

1. **Transfer Functions:** Link LTI System Descriptions:
   a. Impulse response $\delta[n] \rightarrow h[n]$
   b. Input-output pair $x[n] \rightarrow y[n]$
   c. ARMA difference equations
   d. Poles and zeros and PZ diagrams
2. **Inverse Systems:** Undo a system
   a. Minimum phase systems
   b. Non-minimum-phase: Need 2-sided $\mathcal{Z}^{-1}$

TRANSFER FUNCTION [1/4]: IMPULSE RESPONSE h[n]

DEF: The *transfer* or *system* function $H(z)$ of an LTI system is the z-transform of its impulse response: $H(z) = \mathcal{Z}\{h[n]\}$.

Note: This is ZSR (Zero State Response); Initial conditions are all set to zero.

TRANSFER FUNCTION [2/4]: INPUT-OUTPUT PAIR x[n]→y[n]

Note: If a **specific** input $x[n]$ leads to a **specific** output $y[n]$

Then: $y[n] = h[n] * x[n]$. Take $\mathcal{Z}$:

Get: $Y(z)=H(z)X(z) \rightarrow H(z) = \frac{Y(z)}{X(z)}$.

So: $H(z) = \mathcal{Z}\{y[n]\}/\mathcal{Z}\{x[n]\}$.

TRANSFER FUNCTION [3/4]: ARMA DIFFERENCE EQUATIONS

ARMA: $\sum_{i=0}^{N} a_i y[n-i]=\sum_{j=0}^{M} b_j x[n-j]$.

Take $\mathcal{Z}$: $\sum_{i=0}^{N} a_i z^{-i} Y(z)=\sum_{j=0}^{M} b_j z^{-j} X(z)$.

Then: $H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{j=0}^{M} b_j z^{-j}}{\sum_{i=0}^{N} a_i z^{-i}} = \frac{\sum_{j=0}^{M} b_j z^{M-j}}{\sum_{i=0}^{N} a_i z^{N-i}} z^{N-M}$

Note: Don't try to plug into this formula–take $\mathcal{Z}$.

TRANSFER FUNCTION [4/4]: POLES AND ZEROS AND PZ DIAGRAMS

Given: $H(z)=C\frac{z^M+b_1 z^{M-1}+\ldots+b_M}{z^N+a_1 z^{N-1}+\ldots+a_N}$=ratio of 2 polynomials.

Poles: Roots of $z^N+a_1 z^{N-1}+\ldots+a_N=0$. Designated by "X"
Zeros: Roots of $z^M+b_1 z^{M-1}+\ldots+b_M=0$. Designated by "O"

Note: If know poles $\{p_i\}$ and zeros $\{z_i\}$,
Then: $H(z)=C\frac{(z-z_1)\ldots(z-z_M)}{(z-p_1)\ldots(z-p_N)}$ to a scale factor C.
Need: Additional information: Get C from $H(0)$.

Go from any description to any other description by: First computing H(z), then computing anything else:

I/O: Input $x[n] \rightarrow \boxed{\mathbf{h[n]}} \rightarrow$output $y[n]$ $\underbrace{\Longleftrightarrow}_{Y(z)/X(z)}$ $\mathbf{H(z)}$ $\underbrace{\Longleftrightarrow}_{\mathcal{Z}\{h[n]\}}$ $h[n]$

ARMA: $\sum a[i]y[n-i]=\sum b[j]x[n-j]$ $\underbrace{\Longleftrightarrow}_{B(z)/A(z)}$ $\mathbf{H(z)}$ $\underbrace{\Longleftrightarrow}_{C\prod\frac{z-z_i}{z-p_i}}$ **POLES ZEROS**

"Whether you're going to heaven or to hell, you're going to have to transfer in Atlanta."

EXAMPLE #1 [1/2]

Given: $x[n]=(-2)^n u[n] \rightarrow \boxed{\text{LTI}} \rightarrow y[n]=\frac{2}{3}(-2)^n u[n]+\frac{1}{3}u[n]$.

Note: $y[n]=\begin{smallmatrix}\text{FORCED}\\\text{RESPONSE}\end{smallmatrix}\left[\begin{smallmatrix}\text{like}\\x[n]\end{smallmatrix}\right]+\begin{smallmatrix}\text{NATURAL}\\\text{RESPONSE}\end{smallmatrix}\left[\begin{smallmatrix}\text{like}\\h[n]\end{smallmatrix}\right]$.

Goal #1: Compute impulse response $h[n]$.
Goal #2: Get ARMA difference equation.

EXAMPLE #1 [2/2]

Soln: $Y(z)=\frac{2}{3}\frac{z}{z+2}\frac{z-1}{z-1}+\frac{1}{3}\frac{z}{z-1}\frac{z+2}{z+2}=\frac{z^2}{(z+2)(z-1)}$.
And: $X(z)=\frac{z}{z+2} \rightarrow H(z)=\frac{Y(z)}{X(z)}=\frac{z^2}{(z+2)(z-1)}/\frac{z}{z+2}$.
Get: $H(z)=\frac{z}{z-1} \rightarrow h[n]=u[n]$. Like natural response.

ARMA: $H(z)=\frac{Y(z)}{X(z)}=\frac{z}{z-1}\frac{z^{-1}}{z^{-1}}=\frac{1}{1-z^{-1}}$. Cross-multiply:
Get: $Y(z)(1-z^{-1})=X(z)1$. $\mathcal{Z}^{-1} \rightarrow y[n]-y[n-1]=x[n]$.
OR: $Y(z)(z-1)=X(z)z$. $\mathcal{Z}^{-1} \rightarrow y[n+1]-y[n]=x[n+1]$.

EXAMPLE #2 [1/2]

Given: LTI system has: zero at 1; pole at 3; $H(0)=1$.
Goal #1: Compute $h[n]$ and step response (to input $u[n]$).
Goal #2: Compute ARMA difference equation.

Soln: $H(z)=C\frac{z-1}{z-3}$. Then $1=H(0)=C\frac{0-1}{0-3} \rightarrow C=3$.
So: $H(z)=3\frac{z-1}{z-3}$. $h[n]=3(3)^n u[n]-3(3)^{n-1}u[n-1]$
since: $H(z)=3\frac{z}{z-3}-3z^{-1}\frac{z}{z-3}$ and z^{-1} delays by 1.

EXAMPLE #2 [2/2]

Step: $U(z)=\frac{z}{z-1} \rightarrow Y(z)=H(z)U(z)=3\frac{z-1}{z-3}\frac{z}{z-1}=3\frac{z}{z-3}$
response $y[n]=3(3)^n u[n]$=step response of system.
ARMA: $H(z)=\frac{Y(z)}{X(z)}=3\frac{z-1}{z-3}\frac{z^{-1}}{z^{-1}}=3\frac{1-z^{-1}}{1-3z^{-1}}$. Cross-multiply:
Get: $Y(z)(1-3z^{-1})=X(z)(3-3z^{-1}) \rightarrow y[n]-3y[n-1]=3x[n]-3x[n-1]$.

EXAMPLE #3 [1/2]

Given: $x[n] \rightarrow \boxed{y[n]-2y[n-1]=x[n-1]-x[n-2]} \rightarrow y[n]$
Goal: If input $x[n]=3^n u[n]$, compute output $y[n]$.

Soln: Take $\mathcal{Z} \rightarrow Y(z)(1-2z^{-1})=X(z)(z^{-1}-z^{-2})$.
Then: $H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}-z^{-2}}{1-2z^{-1}}\frac{z^2}{z^2}=\frac{z-1}{z^2-2z}$.
And: Zeros: $\{1\}$. Poles: $\{0,2\}$. Also, $X(z)=\frac{z}{z-3}$.

EXAMPLE #3 [2/2]

Get: $Y(z)=H(z)X(z)=\frac{z-1}{z(z-2)}\frac{z}{z-3}=\frac{z-1}{(z-2)(z-3)}=\frac{2}{z-3}-\frac{1}{z-2}$.
$\mathcal{Z}^{-1}$: $y[n]=2(3)^{n-1}u[n-1]-(2)^{n-1}u[n-1]$. Note $y[0]=0$.
OR: $\frac{Y(z)}{z}=\frac{2/3}{z-3}-\frac{1/2}{z-2}-\frac{1/6}{z} \rightarrow Y(z)=\frac{2}{3}\frac{z}{z-3}-\frac{1}{2}\frac{z}{z-2}-\frac{1}{6}\frac{z}{z}$.
Get: $y[n]=\frac{2}{3}(3)^n u[n]-\frac{1}{2}(2)^n u[n]-\frac{1}{6}\delta[n]$. Agrees (try it).
Note: $y[n]=\begin{smallmatrix}\text{FORCED}\\\text{RESPONSE}\end{smallmatrix}\left[\begin{smallmatrix}\text{like}\\x[n]\end{smallmatrix}\right]+\begin{smallmatrix}\text{NATURAL}\\\text{RESPONSE}\end{smallmatrix}\left[\begin{smallmatrix}\text{like}\\h[n]\end{smallmatrix}\right]$.

EXAMPLE #4 [1/3]

Given: $x[n] \rightarrow \boxed{y[n] = x[n] - \frac{3}{4}x[n-1] + \frac{1}{8}x[n-2]} \rightarrow y[n]$

Huh? $x[n]$=transmitted cell phone signal.

And: $y[n]$=received cell phone signal due to reflections off of buildings (multipath).

Goal: Compute **inverse system** that recovers $x[n]$ from $y[n]$.

Huh? $x[n] \rightarrow \boxed{\mathbf{h[n]}} \rightarrow y[n] \rightarrow \boxed{\mathbf{g[n]}} \rightarrow x[n]$.

That is: System $g[n]$ *undoes* the effects of system $h[n]$.

EXAMPLE #4 [2/3]

Idea: Systems in cascade (series): Convolve impulse responses.

Want: Overall impulse response $h[n] * g[n]=\delta[n]$.

Take $\mathcal{Z}$: $h[n] * g[n]=\delta[n] \rightarrow H(z)G(z)=1 \rightarrow G(z)=\frac{1}{H(z)}$.

Soln: Read off $h[n]=\{\underline{1}, -\frac{3}{4}, \frac{1}{8}\}$ from the MA system

$x[n] \rightarrow \boxed{y[n] = x[n] - \frac{3}{4}x[n-1] + \frac{1}{8}x[n-2]} \rightarrow y[n]$

Then: $H(z)=\mathcal{Z}\{h[n]\}=1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}=\frac{z^2-\frac{3}{4}z+\frac{1}{8}}{z^2}$.

EXAMPLE #4 [3/3]

Have: $H(z)=\mathcal{Z}\{h[n]\}=1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}=\frac{z^2-\frac{3}{4}z+\frac{1}{8}}{z^2}$.

Then: $G(z)=\frac{1}{H(z)}=\frac{z^2}{z^2-\frac{3}{4}z+\frac{1}{8}}=\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$.

$\frac{G(z)}{z} = \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})}=\frac{2}{z-\frac{1}{2}}-\frac{1}{z-\frac{1}{4}} \rightarrow G(z)=2\frac{z}{z-\frac{1}{2}}-\frac{z}{z-\frac{1}{4}}$.

g[n]: $2(\frac{1}{2})^n u[n]-(\frac{1}{4})^n u[n]$. Stable and causal inverse system.

MINIMUM PHASE SYSTEMS [1/2]

g[n]: **Stable and causal** since all poles of $G(z)$ inside $|z|$=1.

Note: Poles of $G(z)$=Zeros of $H(z)$. Need zeros of $H(z)$ inside $|z|$=1.

DEF: $H(z)$ is a *minimum phase* system

means: All poles & zeros of $H(z)$ inside $|z|$=1.

Point: Minimum phase systems have stable and causal inverse systems.

Note: $H(z)$ is minimum phase$\rightarrow \frac{1}{H(z)}$ also minimum phase.

MINIMUM PHASE SYSTEMS [2/2]

If: System $H(z)$ is a minimum phase system

Then: $G(z)=\frac{1}{H(z)}$ has a stable and causal $g[n]$.

If: System $H(z)$ is NOT a minimum phase system

Then: $G(z)=\frac{1}{H(z)}$ has a stable but **two-sided** $g[n]$.

If: $H(z)$ has no zeros ON unit circle $|z|$=1.

So? What good is a non-causal inverse system? See below.

EXAMPLE #5 [1/5]

Given: $x[n] \rightarrow \boxed{y[n] = x[n] - 2.5x[n-1] + x[n-2]} \rightarrow y[n]$

Goal: Compute **inverse system**: recovers $x[n]$ from $y[n]$.

Huh? $x[n] \rightarrow \boxed{\mathbf{h[n]}} \rightarrow y[n] \rightarrow \boxed{\mathbf{g[n]}} \rightarrow x[n]$.

Soln: Read off $h[n]=\{\underline{1}, -2.5, 1\}$ from the MA system.

Then: $H(z)=\mathcal{Z}\{h[n]\}=1-2.5z^{-1}+z^{-2}=\frac{z^2-2.5z+1}{z^2}$.

EXAMPLE #5 [2/5]

Have: $H(z)=\mathcal{Z}\{h[n]\}=1-2.5z^{-1}+z^{-2}=\frac{z^2-2.5z+1}{z^2}$.

Then: $G(z)=\frac{1}{H(z)}=\frac{z^2}{z^2-2.5z+1}=\frac{z^2}{(z-2)(z-\frac{1}{2})}$.

$\frac{G(z)}{z}=\frac{z}{(z-2)(z-\frac{1}{2})}=\frac{4}{3}\frac{1}{z-2}-\frac{1}{3}\frac{1}{z-\frac{1}{2}} \rightarrow G(z)=\frac{4}{3}\frac{z}{z-2}-\frac{1}{3}\frac{z}{z-\frac{1}{2}}$.

g[n]: $\frac{4}{3}(2)^n u[n]-\frac{1}{3}(\frac{1}{2})^n u[n]$. Causal but unstable inverse system!

Why? $H(z)$ is **not** minimum phase→no causal and stable $g[n]$.

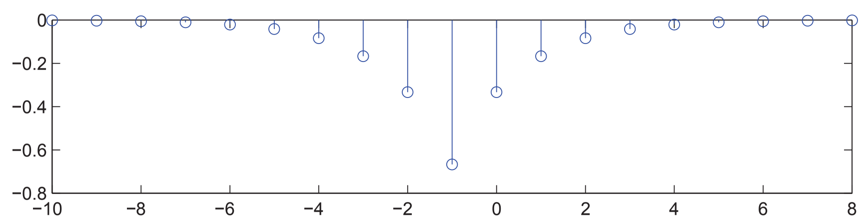
EXAMPLE #5 [3/5]

Need: 2-sided but stable inverse z-transform, which is:
g[n]: $-\frac{4}{3}(2)^n u[-n-1]-\frac{1}{3}(\frac{1}{2})^n u[n]=-\frac{2}{3}(\frac{1}{2})^{|n+1|}$ (try it).
But: How can we implement a non-causal system?!

Idea: $g[n]$ is negligible outside $-10\leq n\leq 8$ since $2^{-10} < 0.001$.
So: Truncate $g[n]$ to the interval $-10\leq n\leq 8$ (see next slide)
by: Setting $g[n]=0$ outside the interval $-10\leq n\leq 8$.
Then: Use the causal & stable (since MA) system $g[n-10]$.

EXAMPLE #5 [4/5]

g[n]: $-\frac{4}{3}(2)^n u[-n-1]-\frac{1}{3}(\frac{1}{2})^n u[n]=-\frac{2}{3}(\frac{1}{2})^{|n+1|}$:
Matlab: `>> N=[-10:8];stem(N,-2/3*(1/2).^(abs(N+1)))`



EXAMPLE #5 [5/5]

Soln: Use the causal & stable MA system with impulse response
$g[n-10]= -\frac{2}{3}\{\underline{(\frac{1}{2})^9}, (\frac{1}{2})^8 \ldots 1 \ldots (\frac{1}{2})^9\}$. Duration=19.

Then: $x[n] \rightarrow \boxed{\mathbf{h[n]}} \rightarrow y[n] \rightarrow \boxed{\mathbf{g[n-10]}} \rightarrow x[n-10]$.

That is: Can't recover $x[n]$, but CAN recover $x[n-10]$.
And: Sampling at $44100\frac{\text{SAMPLE}}{\text{SECOND}}$, recover $x(t-\frac{1}{4410})$.

Use: $x[n-10]=-\frac{2}{3}(\frac{1}{2})^9 y[n]-\frac{2}{3}(\frac{1}{2})^8 y[n-1]-\ldots-\frac{2}{3}(\frac{1}{2})^9 y[n-18]$

Not: $x[n]=-\frac{2}{3}(\frac{1}{2})^9 y[n+10]-\frac{2}{3}(\frac{1}{2})^8 y[n+9]-\ldots-\frac{2}{3}(\frac{1}{2})^9 y[n-8]$.

POLES vs. MODES [1/2]

Poles: Roots of denominator=0.
Modes: Roots of denominator=0.
Huh? Poles can be cancelled by zeros.
But: Modes can't be cancelled by zeros.
Why? Modes associated with ZIR.
But: Poles associated with ZSR.

POLES vs. MODES [2/2]

EX: $y[n]-3y[n-1]+2y[n-2]=x[n]-x[n-1]$.

Then: $H(z)=\frac{1-z^{-1}}{1-3z^{-1}+2z^{-2}}\frac{z^2}{z^2}=\frac{z(z-1)}{(z-2)(z-1)}$.

Modes: $\{1,2\}$ from denominator.
Poles: $\{2\}$, since $\{1\}$ cancelled.

Note: Unrealistic-roundoff error!