TOPICS FOR TODAY’S LECTURE

1. Transfer Functions: Link LTI System Descriptions:
   a. Impulse response \( \delta[n] \rightarrow h[n] \)
   b. Input-output pair \( x[n] \rightarrow y[n] \)
   c. ARMA difference equations
   d. Poles and zeros and PZ diagrams

2. Inverse Systems: Undo a system
   a. Minimum phase systems
   b. Non-minimum-phase: Need 2-sided \( Z^{-1} \)

TRANSFER FUNCTION [1/4]:
IMPULSE RESPONSE \( h[n] \)

DEF: The transfer or system function \( H(z) \) of an LTI system is the \( z \)-transform of its impulse response: \( H(z) = Z\{h[n]\} \).

Note: This is ZSR (Zero State Response); Initial conditions are all set to zero.

TRANSFER FUNCTION [2/4]:
INPUT-OUTPUT PAIR \( x[n] \rightarrow y[n] \)

Note: If a specific input \( x[n] \)
leads to a specific output \( y[n] \)

Then: \( y[n] = h[n] \ast x[n] \). Take \( Z \):
Get: \( Y(z) = H(z)X(z) \rightarrow H(z) = \frac{Y(z)}{X(z)} \).
So: \( H(z) = Z\{y[n]\} / Z\{x[n]\} \).

TRANSFER FUNCTION [3/4]:
ARMA DIFFERENCE EQUATIONS

ARMA: \( \sum_{i=0}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j] \).

Take \( Z \): \( \sum_{i=0}^{N} a_i z^{-i} Y(z) = \sum_{j=0}^{M} b_j z^{-j} X(z) \).

Then: \( H(z) = \frac{Y(z)}{X(z)} = \sum_{i=0}^{N} a_i z^{-i} = \sum_{j=0}^{M} b_j z^{-j} z^{N-M} \).

Note: Don’t try to plug into this formula—take \( Z \).

TRANSFER FUNCTION [4/4]:
POLES AND ZEROS AND PZ DIAGRAMS

Given: \( H(z) = \frac{C z^M + b_1 z^{M-1} + \ldots + b_M}{z^N + a_1 z^{N-1} + \ldots + a_N} \) = ratio of 2 polynomials.

Poles: Roots of \( z^N + a_1 z^{N-1} + \ldots + a_N = 0 \). Designated by “X”
Zeros: Roots of \( z^M + b_1 z^{M-1} + \ldots + b_M = 0 \). Designated by “O”

Note: If know poles \( \{p_i\} \) and zeros \( \{z_i\} \),
Then: \( H(z) = C \frac{(z-p_1) \ldots (z-p_N)}{(z-z_1) \ldots (z-z_M)} \) to a scale factor \( C \).
Need: Additional information: Get \( C \) from \( H(0) \).

Go from any description to any other description by:
First computing \( H(z) \), then computing anything else:

I/O: Input \( x[n] \) \( \rightarrow \) output \( y[n] \) \( \overset{\leftrightarrow}{\iff} \) \( H(z) \) \( \iff \) \( h[n] \)

ARMA: \( \sum a[i] y[n-i] = \sum b[j] x[n-j] \) \( \iff \) \( H(z) \) \( \iff \) \( h[n] \)

POLES ZEROS

“Whether you’re going to heaven or to hell, you’re going to have to transfer in Atlanta.”
EXAMPLE #1 [1/2]

Given: \( x[n] = (-2)^n u[n] \) \( \rightarrow \) \( y[n] = \frac{2}{3} (-2)^n u[n] + \frac{1}{2} u[n] \).

Note: \( y[n] = \text{FORCED RESPONSE} \left[ \frac{2}{3} \right] + \text{NATURAL RESPONSE} \left[ \frac{1}{2} \right] \).

Goal #1: Compute impulse response \( h[n] \).
Goal #2: Get ARMA difference equation.

EXAMPLE #2 [1/2]

Given: LTI system has: zero at 1; pole at 3; \( H(0) = 1 \).
Goal #1: Compute \( h[n] \) and step response (to input \( u[n] \)).
Goal #2: Compute ARMA difference equation.

Soln: \( H(z) = C \frac{z-1}{z^3} \). Then \( 1 = H(0) = C \frac{0 - 1}{0 - 3} \rightarrow C = 3 \).
So: \( H(z) = 3 \frac{z-1}{z^3} \). \( h[n] = 3(3)^n u[n] - 3(3)^{n-1} u[n-1] \)
since: \( H(z) = 3 \frac{z-1}{z^3} - 3z^{-1} \frac{z-1}{z^2} \) and \( z^{-1} \) delays by 1.

EXAMPLE #3 [1/2]

Given: \( x[n] \rightarrow \frac{y[n] - 2y[n-1]}{3} = x[n-1] - x[n-2] \rightarrow y[n] \)

Goal: If input \( x[n] = 3^n u[n] \), compute output \( y[n] \).

Soln: Take \( Z \rightarrow Y(z)(1 - 2z^{-1}) = X(z)(z^{-1} - z^{-2}) \).
Then: \( H(z) = \frac{Y(z)}{X(z)} = z^{-1} - z^{-2} \)
And: Zeros: \( \{1\} \). Poles: \( \{0, 2\} \). Also, \( X(z) = \frac{z}{z^2 - 3} \).

EXAMPLE #1 [2/2]

Soln: \( Y(z) = \frac{2}{3} + \frac{1}{3} \frac{2}{3} \frac{z}{3} + \frac{1}{3} \frac{z^2}{3^2} = (z+2)(z+1)(z-1) \).
And: \( X(z) = \frac{z}{z+1} \rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{2}{z+2} \frac{z}{z+1} / (z+2) \frac{z}{z+1} \).
Get: \( H(z) = \frac{z}{z+2} \rightarrow h[n] = u[n] \). Like natural response.

ARMA: \( H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z+1} \frac{z^{-1}}{z+1} = \frac{1}{z-1} \). Cross-multiply:
Get: \( Y(z)(1-z^{-1}) = X(z)1 \). \( Z^{-1} \rightarrow y[n] - y[n-1] = x[n] \).
OR: \( Y(z)(z^{-1}) = X(z)z \). \( Z^{-1} \rightarrow y[n+1] - y[n] = x[n+1] \).

EXAMPLE #2 [2/2]

Step: \( U(z) = \frac{z}{z+1} \rightarrow Y(z) = H(z)U(z) = 3 \frac{z^{-1}}{z^2 + 3z - 1} = 3 \frac{z}{z^2 + 3} \)
response \( y[n] = 3(3)^n u[n] \) = step response of system.

ARMA: \( H(z) = \frac{Y(z)}{X(z)} = 3 \frac{z^{-1}}{z^2 + 3z - 1} = 3 \frac{z}{3z + 1} \). Cross-multiply:
Get: \( Y(z)(1 - 3z^{-1}) = X(z)(3 - 3z^{-1}) \rightarrow y[n] - 3y[n-1] = 3x[n] - 3x[n-1] \).

EXAMPLE #3 [2/2]

Get: \( Y(z) = H(z)X(z) = \frac{z-1}{z^2 + 3} \frac{z^{-1}}{z^2 + 3} = \frac{z^{-1}}{z^2 + 3} \frac{z^{-1}}{z^2 + 3} = \frac{z^{-1}}{z^2 + 3} \).
\( Z^{-1}: \ y[n] = 2(3)^n u[n-1] - \frac{1}{2} (2)^n u[n-1] \). Note \( y[0] = 0 \).
OR: \( Y(z) = \frac{2}{z^2 - 3} \frac{1}{z + 2} \frac{1}{z + 2} \rightarrow Y(z) = \frac{2}{z^2 - 3} \frac{z}{z + 2} \frac{1}{z + 2} \).
Get: \( y[n] = \frac{3}{3} (3)^n u[n] - \frac{1}{2} (2)^n u[n] - \frac{1}{6} \delta[n] \). Agrees (try it).

Note: \( y[n] = \text{FORCED RESPONSE} \left[ \frac{2}{3} \right] + \text{NATURAL RESPONSE} \left[ \frac{1}{6} \right] \).
EXAMPLE #4 [1/3]

Given: \( x[n] - y[n] = x[n] - \frac{3}{5}x[n-1] + \frac{1}{5}x[n-2] \rightarrow y[n] \)

Huh? \( x[n] \)= transmitted cell phone signal.

And: \( y[n] \)= received cell phone signal due to reflections off of buildings (multipath).

Goal: Compute inverse system that recovers \( x[n] \) from \( y[n] \).

Huh? \( x[n] - h[n] \rightarrow y[n] - g[n] \rightarrow x[n] \).

That is: System \( g[n] \) undoes the effects of system \( h[n] \).

EXAMPLE #4 [2/3]


Want: Overall impulse response \( h[n] \ast g[n] = \delta[n] \).

Take \( Z \): \( h[n] \ast g[n] = \delta[n] \rightarrow H(z)G(z) = 1 \rightarrow G(z) = \frac{1}{H(z)} \).

Soln: Read off \( h[n] = \{1, -\frac{3}{5}, \frac{1}{5}\} \) from the MA system

\( x[n] \rightarrow y[n] = x[n] - \frac{3}{5}x[n-1] + \frac{1}{5}x[n-2] \rightarrow y[n] \)

Then: \( H(z) = Z\{h[n]\} = 1 - \frac{3}{5}z^{-1} + \frac{1}{5}z^{-2} = \frac{z^2 - \frac{6}{5}z + 1}{z^2 - \frac{1}{5}} \).

EXAMPLE #4 [3/3]

Have: \( H(z) = Z\{h[n]\} = 1 - \frac{3}{5}z^{-1} + \frac{1}{5}z^{-2} = \frac{z^2 - \frac{6}{5}z + 1}{z^2 - \frac{1}{5}} \).

Then: \( G(z) = \frac{1}{H(z)} = \frac{z^2 - \frac{6}{5}z + 1}{z^2 - \frac{1}{5}} \).

\[ \frac{G(z)}{z} = \frac{z}{(z-\frac{3}{5})(z-\frac{1}{5})} \cdot \frac{2}{z^{-1}} \cdot \frac{2}{z^{-3}} \rightarrow G(z) = 2z^{-1} + \frac{3}{5}z^{-3} \]

\( g[n] = 2\left(\frac{1}{5}\right)^n u[n] - \left(\frac{3}{5}\right)^n u[n] \). Stable and causal inverse system.

MINIMUM PHASE SYSTEMS [1/2]

\( g[n] \): Stable and causal since all poles of \( G(z) \) inside \( |z|=1 \).

Note: Poles of \( G(z) = \text{Zeros of } H(z) \). Need zeros of \( H(z) \) inside \( |z|=1 \).

DEF: \( H(z) \) is a minimum phase system

means: All poles & zeros of \( H(z) \) inside \( |z|=1 \).

Point: Minimum phase systems have stable and causal inverse systems.

Note: \( H(z) \) is minimum phase \( \rightarrow \frac{1}{H(z)} \) also minimum phase.

MINIMUM PHASE SYSTEMS [2/2]

If: System \( H(z) \) is a minimum phase system

Then: \( G(z) = \frac{1}{H(z)} \) has a stable and causal \( g[n] \).

If: System \( H(z) \) is NOT a minimum phase system

Then: \( G(z) = \frac{1}{H(z)} \) has a stable but two-sided \( g[n] \).

If: \( H(z) \) has no zeros on unit circle \( |z|=1 \).

So? What good is a non-causal inverse system? See below.

EXAMPLE #5 [1/5]

Given: \( x[n] - y[n] = x[n] - 2.5x[n-1] + x[n-2] \rightarrow y[n] \)

Goal: Compute inverse system: recovers \( x[n] \) from \( y[n] \).

Huh? \( x[n] - h[n] \rightarrow y[n] - g[n] \rightarrow x[n] \).

Soln: Read off \( h[n] = \{1, -2.5, 1\} \) from the MA system.

Then: \( H(z) = Z\{h[n]\} = 1 - 2.5z^{-1} + z^{-2} = \frac{z^2 - 2.5z + 1}{z^2} \).
EXAMPLE #5 [2/5]

Have: \[ H(z) = \mathcal{Z}\{h[n]\} = 1 - 2.5z^{-1} + z^{-2} = \frac{z^2 - 2.5z + 1}{z^2}. \]

Then: \[ G(z) = \frac{1}{H(z)} = \frac{z^2}{z^2 - 2.5z + 1} = \frac{\frac{z}{z^2 - \frac{5}{2}z + 1}}{\frac{1}{z}}. \]

\[ \frac{G(z)}{z} = \frac{\frac{z}{z^2 - \frac{5}{2}z + 1}}{\frac{1}{z}} \rightarrow G(z) = \frac{\frac{z}{z^2 - \frac{5}{2}z + 1}}{\frac{1}{z}}. \]

\[ g[n] = \frac{3}{4}(2)^nu[n] - \frac{1}{4}(\frac{1}{2})^nu[n]. \] Causal but unstable inverse system!

Why? \( H(z) \) is not minimum phase → no causal and stable \( g[n] \).

EXAMPLE #5 [3/5]

Need: 2-sided but stable inverse z-transform, which is:
\[ g[n] = -\frac{3}{4}(2)^nu[n] - \frac{1}{4}(\frac{1}{2})^nu[n] = -\frac{3}{4}(\frac{1}{2})^nu[n]. \] (try it).

But: How can we implement a non-causal system?!

Idea: \( g[n] \) is negligible outside \(-10 \leq n \leq 8\) since \( 2^{-10} < 0.001 \).

So: Truncate \( g[n] \) to the interval \(-10 \leq n \leq 8\) (see next slide)
by: Setting \( g[n] = 0 \) outside the interval \(-10 \leq n \leq 8\).

Then: Use the causal & stable (since MA) system \( g[n-10] \).

EXAMPLE #5 [4/5]

\[ g[n] = -\frac{3}{4}(2)^nu[n] - \frac{1}{4}(\frac{1}{2})^nu[n] = -\frac{3}{4}(\frac{1}{2})^nu[n]. \]

Matlab: \[ >> N=[-10:8]; stem(N,-2/3*(1/2)\.ˆ(abs(N+1))) \]

That is: Can’t recover \( x[n] \), but CAN recover \( x[n-10] \).

And: Sampling at \( N = 44100 \) sampling rate, recover \( x(t - \frac{44100}{10}) \).

Use: \[ x[n-10] = -\frac{3}{4}(\frac{1}{2})^9y[n-10] - \frac{1}{4}(\frac{1}{2})^8y[n-9] - \cdots - \frac{1}{4}(\frac{1}{2})^3y[n-3] \]

Not: \[ x[n] = -\frac{3}{4}(\frac{1}{2})^9y[n+10] - \frac{1}{4}(\frac{1}{2})^8y[n+9] - \cdots - \frac{1}{4}(\frac{1}{2})^3y[n+3] \]

EXAMPLE #5 [5/5]

Soln: Use the causal & stable MA system with impulse response \[ g[n]= -\frac{3}{4}(\frac{1}{2})^9, (\frac{1}{2})^8 \ldots 1 \ldots (\frac{1}{2})^0. \] Duration=19.

Then: \[ x[n-10] \rightarrow [h[n] \rightarrow y[n] \rightarrow [g[n-10] \rightarrow x[n-10]] \]

That is: Can’t recover \( x[n] \), but CAN recover \( x[n-10] \).

POLES vs. MODES [1/2]

POLES: Roots of denominator=0.

Modes: Roots of denominator=0.

Huh? Poles can be cancelled by zeros.

But: Modes can’t be cancelled by zeros.

Why? Modes associated with ZIR.

But: Poles associated with ZSR.

EX: \( y[n-3y[n-1]+2y[n-2] = x[n]-x[n-1] \).

Then: \( H(z) = \frac{1-\frac{3}{4}z^{-1}+\frac{1}{4}z^{-2}}{(z-1)(z-\frac{1}{2})}. \)

POLES vs. MODES [2/2]

POLES: \{1, 2\} from denominator.

Poles: \{2\}, since \{1\} cancelled.

Note: Unrealistic-roundoff error!