SAMPLING THEOREM

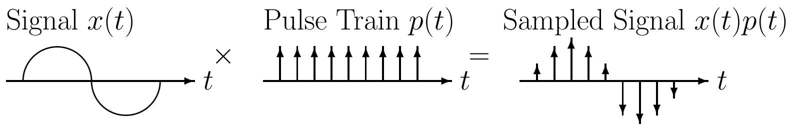
1. Statement of Sampling Theorem
2. Derivation of Sampling Theorem
3. Ideal Reconstruction from Samples
4. Determining Signal Bandwidths
5. Finite Pulse Width Sampling
6. Undersampling and Aliasing

SAMPLING THEOREM: STATEMENT [1/3]

- **Given:** Continuous-time signal $x(t)$.
- **That's:** Bandlimited to B Hertz.
- **Means:** Maximum frequency is B Hertz.
- **Means:** $X(\omega) = \mathcal{F}\{x(t)\}=0$ for $|\omega| \geq 2\pi B$.
- **Means:** Bandwidth=B Hertz=$2\pi B \frac{\text{RADIAN}}{\text{SECOND}}$.

SAMPLING THEOREM: STATEMENT [2/3]

- **Given:** Knowledge of only the *samples* $\{x(nT)\}$ of $x(t)$.
- **Means:** $x(t)$ *sampled* every T seconds, at *rate* $\boxed{S=\frac{1}{T}\frac{\text{SAMPLE}}{\text{SECOND}}}$.



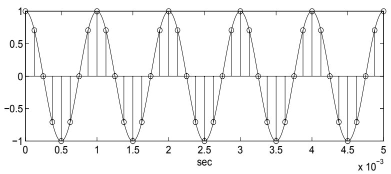
SAMPLING THEOREM: STATEMENT [3/3]

- **Then:** $x(t)$ can be reconstructed from its samples $\{x(nT)\}$
- **If:** Sampling rate $S = \frac{1}{T}\frac{\text{SAMPLE}}{\text{SECOND}} > 2B$=2(bandwidth).
- **Where:** $\boxed{S > 2B}$ Here $2B$ is the *Nyquist* sampling rate.
- **Note:** Co-discovered by Claude Shannon (UM Class of 1938)
- **Note:** Digital Signal Processing is possible because of this.

SAMPLING THEOREM: EXAMPLE #1

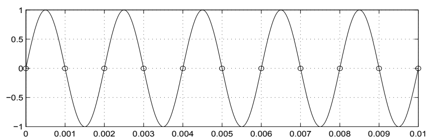
$x(t)$=$\cos(2\pi 1000t)$ sampled at S=$8000\frac{\text{SAMPLE}}{\text{SECOND}}$. T=$\frac{1}{S}$=$\frac{1}{8000}$ sec.
Set t=nT=$n/8000$: $x(nT)$=$\cos(2\pi\frac{1000}{8000}n)$=$\cos(\frac{\pi}{4}n)$.
Can reconstruct $x(t)$ from its samples $\{x(nT)\}$: (How? See below.)

n	...	0	1	2	3	4	...
$x(nT)$	...	1.0	.71	0.0	–.71	–1.0	...



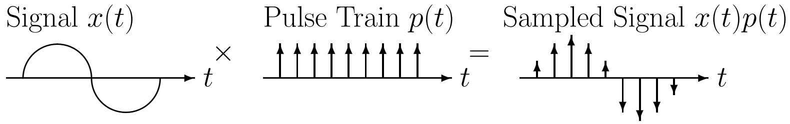
SAMPLING THEOREM: EXAMPLE #2

Do we need $S > 2B$ or only $S \geq 2B$?
$x(t)$=$\sin(2\pi 500t)$ sampled at S=$1000\frac{\text{SAMPLE}}{\text{SECOND}}$. T=$\frac{1}{S}$=$\frac{1}{1000}$ sec.
Set t=nT=$n/1000$: $x(nT)$=$\sin(2\pi\frac{500}{1000}n)$=$\sin(\pi n)$=0!
Can't reconstruct $x(t)$ from its samples $\{x(nT)\}$!



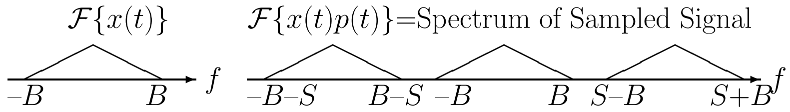
SAMPLING THEOREM: PROOF [1/3]

- **DEF:** $p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$=train (series) of impulses.
- **DEF:** $x(t)p(t)$=*sampled signal*=train of impulses weighted by $\{x(nT)\}$.
- **Note:** $x(t)p(t)=x(t)\sum\delta(t-nT)=\sum x(nT)\delta(t-nT)$.
- **So:** Only need samples $\{x(nT)\}$ to create $x(t)p(t)$.



SAMPLING THEOREM: PROOF [2/3]

- **Fourier series expansion:** $p(t)=\sum p_k e^{j2\pi kt/T}=\frac{1}{T}\sum e^{j2\pi kt/T}$.
- **Fourier coefficient formula:** $p_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-j2\pi kt/T}dt=\frac{1}{T}$.
- **Then:** $\mathcal{F}\{x(t)p(t)\}=\frac{1}{T}\sum\mathcal{F}\{x(t)e^{j2\pi kt/T}\}=\frac{1}{T}\sum X(\omega-2\pi\frac{k}{T})$.
- $S=\frac{1}{T}\rightarrow\mathcal{F}\{x(t)p(t)\}=\frac{1}{T}\sum X(2\pi(f-\frac{k}{T}))=\frac{1}{T}\sum X(2\pi(f-kS))$.

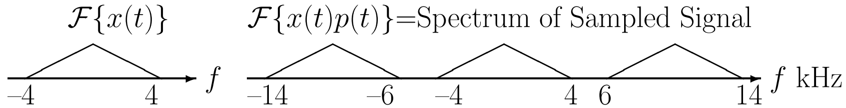


SAMPLING THEOREM: PROOF [3/3]

- **Can:** Reconstruct $x(t)$ from $x(t)p(t)$ if $(S-B)>B\rightarrow\boxed{S>2B}$.
- **By:** Low-pass filtering $x(t)p(t)$. Cutoff frequency=B Hertz.
- **Formula:** $x(t)=\frac{2B}{2B}T\underbrace{\sum x(nT)\delta(t-nT)}_{\text{SAMPLED SIGNAL } x(t)p(t)} * \underbrace{\frac{\sin(2\pi Bt)}{\pi t}}_{\text{LPF } h(t)}$.
- **Interpolation Formula:** $x(t)=\sum x(nT)(2BT)\frac{\sin 2\pi B(t-nT)}{2\pi B(t-nT)}$.

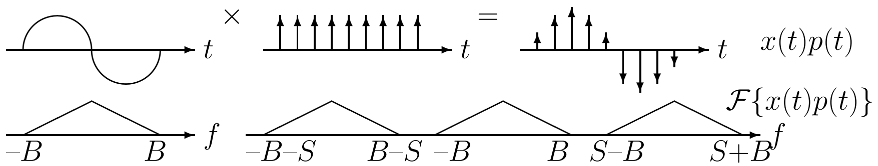
SAMPLING THEOREM: EXAMPLE

- **Given:** Continuous-time $x(t)$ is bandlimited to 4 kHz.
- **Sample:** 10 "kHz"=$10000\frac{\text{SAMPLE}}{\text{SECOND}}>2(4\text{ kHz})$.
- $\mathcal{F}\{x(t)p(t)\}$=Spectrum of sampled signal. Repeats in f!
- **Reconstruct:** Low-pass filter with cutoff=4 kHz.



SAMPLING AND RECONSTRUCTION: SUMMARY

$$\underbrace{x(t)\rightarrow\overset{p(t)}{\overset{\downarrow}{\otimes}}\rightarrow x(t)p(t)\rightarrow\{x(nT)\}}_{\text{SAMPLING}}\underbrace{\rightarrow\overset{p(t)}{\overset{\downarrow}{\otimes}}\rightarrow x(t)p(t)\rightarrow\boxed{\text{LOWPASS FILTER}}\rightarrow x(t)}_{\text{RECONSTRUCTION}}$$



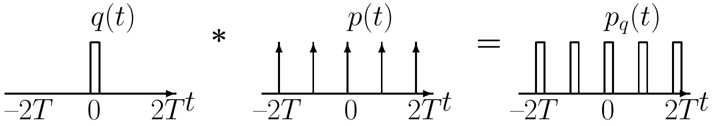
DETERMINING SIGNAL BANDWIDTH: EXAMPLES

1. $x(t)=\underline{\frac{3t+7}{t}\sin(2\pi 5t)}$: Rewrite: $x(t)=3\sin(2\pi 5t)+7\pi\frac{\sin(2\pi 5t)}{\pi t}$.
2. $x(t)=\underline{\sin(2\pi 3t)\sin(2\pi 2t)}$: Rewrite: $x(t)=\frac{1}{2}[\cos(2\pi 1t)-\cos(2\pi 5t)]$.
3. $x(t)=\underline{\frac{\sin(2\pi 3t)\sin(2\pi 2t)}{t^2}}$: Then $X(\omega)=\frac{1}{2\pi}[\text{rect}(\frac{\omega}{6})*\text{rect}(\frac{\omega}{4})]$.
   Convolve length=6 with length=4→length=10.

For all 3: Bandlimited to $\omega=10\pi$ or f=5 Hertz. Need $10\frac{\text{SAMPLE}}{\text{SECOND}}$.

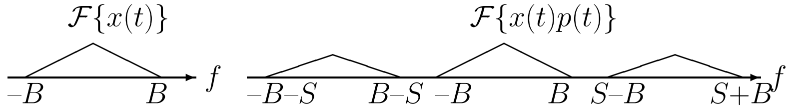
SAMPLING WITH FINITE-WIDTH PULSES [1/2]

- **Problem:** The ideal impulse train $p(t)=\Sigma\,\delta(t-nT)$ doesn't exist!
- **Sol'n:** Use $p_q(t)=\Sigma\,q(t-nT)$ where $q(t)$=short pulse (does exist).
- **Then:** $p_q(t)=\Sigma\,q(t-nT)=\Sigma\,\delta(t-nT)*q(t)=p(t)*q(t)$:



SAMPLING WITH FINITE-WIDTH PULSES [2/2]

- **Fourier series expansion:** $p_q(t)=\Sigma\,p_{q,k}e^{j2\pi kt/T}$. **Pulse width:** Δ.
- **Fourier coefficient formula:** $p_{q,k}=\frac{1}{T}\int_{-\Delta/2}^{\Delta/2} e^{-j2\pi kt/T}dt=\frac{1}{k\pi}\sin(\frac{k\pi\Delta}{T})$.
- **Then:** $\mathcal{F}\{x(t)p_q(t)\}=\Sigma\,p_{q,k}\mathcal{F}\{x(t)e^{j2\pi kt/T}\}=\Sigma\,p_{q,k}X(\omega-2\pi\frac{k}{T})$.
- $S=\frac{1}{T}\rightarrow\mathcal{F}\{x(t)p(t)\}=\Sigma\,p_{q,k}X(2\pi(f-\frac{k}{T}))=\Sigma\,p_{q,k}X(2\pi(f-kS))$.

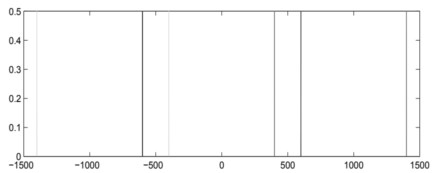


UNDERSAMPLING AND ALIASING [1/4]

- What if $\frac{\text{SAMPLE}}{\text{RATE}}=S<2B=2\frac{\text{MAX.}}{\text{FREQ}}$=Nyquist frequency?
- **Aliasing**: image spectrum masquerades as actual spectrum!
- **EX:** 600 Hz sinusoid sampled at $1000\frac{\text{SAMPLE}}{\text{SECOND}}$. 1000<2(600).
- **Sample:** $x(t)=\cos(2\pi 600t)$ with $T=\frac{1}{1000}$. Set $t=nT=\frac{n}{1000}$:
- **Get:** Samples $x(nT)=\cos(2\pi\frac{600n}{1000})=\cos(1.2\pi n)=\cos(0.8\pi n)$.
- **Since:** $\cos(1.2\pi n)=\cos(1.2\pi n-2\pi n)=\cos(-0.8\pi n)=\cos(0.8\pi n)$.

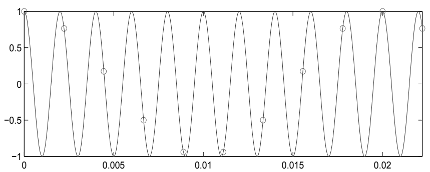
UNDERSAMPLING AND ALIASING [2/4]

- Image spectrum impersonates actual spectrum! Overlap!
- Low-pass filtering this→400 Hz, not 600 Hz, sinusoid!



UNDERSAMPLING AND ALIASING [3/4]

- 500-Hertz sinusoid sampled at $450\frac{\text{SAMPLE}}{\text{SECOND}}$. Samples: circles.
- Did samples come from 450 Hertz or 50 Hertz sinusoid?



UNDERSAMPLING AND ALIASING [4/4]

- $x(t)=\cos(2\pi 300t+1)$ sampled at $500\frac{\text{SAMPLE}}{\text{SECOND}}$. Reconstructed=?
- **Sample:** $t=\frac{n}{500}\rightarrow x(nT)=\cos(2\pi\frac{300}{500}n+1)=\cos(1.2\pi n+1)$.
- **But:** $\cos(1.2\pi n+1)=\cos(1.2\pi n-2\pi n+1)=\cos(-0.8\pi n+1)=\cos(0.8\pi n-1)$.
- **Reconstruct:** $n=500t\rightarrow\cos(0.8\pi(500t)-1)=\boxed{\cos(2\pi 200t-1)}$
- 300 Hertz aliased down to 200 Hertz. Phase also changed.