TOPICS FOR TODAY'S LECTURE

1. Linear Systems:
   a. Scaling Property
   b. Superposition Property
2. Time-Invariant Systems
3. Derivation of Convolution

WHAT IS A SYSTEM?

What: Input $x[n] \rightarrow \boxed{\textbf{SYSTEM}} \rightarrow y[n]$ output.
That is: A system inputs one function $x[n]$
and then outputs another one $y[n]$.
EX: Many examples in a moment.

Cont.: Models circuit, mechanical system.
Discrete: **Design** system to **filter** input $x[n]$:
1. Reduce or eliminate noise or interference
2. Emphasize or de-emphasize signal aspects

DEF: A system is **LINEAR** [1/4]
if these two properties hold:

1. **Scaling:**

If $x[n] \rightarrow \boxed{\textbf{SYSTEM}} \rightarrow y[n]$,
then $ax[n] \rightarrow \boxed{\textbf{SYSTEM}} \rightarrow ay[n]$
for: any **constant** a.
But: NOT true if a varies with time (i.e., $a[n]$).

That is: "Doubling the input doubles the output."

2. **Superposition:** [2/4]

If $x_1[n] \rightarrow \boxed{\textbf{SYSTEM}} \rightarrow y_1[n]$
and $x_2[n] \rightarrow \boxed{\textbf{SYSTEM}} \rightarrow y_2[n]$,
Then: $(ax_1[n] + bx_2[n]) \rightarrow \boxed{\textbf{SYSTEM}} \rightarrow (ay_1[n] + by_2[n])$
for: any **constants** a, b.
But: NOT true if a or b vary with time (i.e., $a[n], b[n]$).

That is: "Response to the sum is the sum of the responses."

EXAMPLES OF LINEAR SYSTEMS [3/4]

$y[n] = 3x[n-2]$; $y[n] = \sin(n)x[n]$.
$y[n] = x[n+1] - nx[n] + 2x[n-1]$.

Q: How to tell whether a system is linear?
A: *If doubling input doubles output,*
then the system is likely linear.
This rule works most (not all) of the time.

EXAMPLES OF Non-LINEAR SYSTEMS [4/4]

$y[n] = x^2[n]$; $y[n] = \sin(x[n])$; $y[n] = |x[n]|$;
Last: Doubling input doubles output, but non-linear.
$y[n] = x[n]/x[n-1]$; $y[n] = x[n] + 1$.
Last: This is an **affine** (linear+constant) system.

Note: Nonlinear function of n is OK.
Nonlinear function of $x[n]$ NOT OK.

DEF: A system is **TIME-INVARIANT** if: [1/3]

If $x[n] \rightarrow \boxed{\textbf{SYSTEM}} \rightarrow y[n]$,
then $x[n-N] \rightarrow \boxed{\textbf{SYSTEM}} \rightarrow y[n-N]$,

for: any integer time delay N.

But: NOT true if N varies with time (e.g., $N(n)$).

That is: "Delaying the input delays the output."

TIME-INVARIANT SYSTEMS [2/3]

$y[n] = 3x[n-2]$; $y[n] = \sin(x[n])$;
$y[n] = x[n]/x[n-1]$ (note this is not linear).

Q: How to tell whether a system is time-invariant?

A: *If "n" occurs* **only** *inside $x[n]$ or $y[n]$, then the system is time-invariant.*

Non-TIME-INVARIANT SYSTEMS [3/3]

$y[n] = nx[n]$; $y[n] = x[n^2]$;
$y[n] = x[2n]$; $y[n] = x[-n]$.

Note: $x[n-3]$ OK for time-invariant;
$x[n^2]$ and $x[-n]$ NOT OK.

LINEAR TIME-INVARIANT SYSTEMS [1/4]

DEF: A system is **Linear Time-Invariant (LTI)** if it is:

1. Linear (scaling and superposition both hold)
2. Time-invariant (what a surprise!)

LTI systems are important in discrete and cont. time:
Linear constant-coefficient differential equations and:

AN IMPORTANT LTI SYSTEM [2/4]

$y[n] + a_1 y[n-1] + \ldots + a_N y[n-N] =$
$b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$
for *constants* $\{a_1 \ldots a_N\}$ and $\{b_0 \ldots b_M\}$.
This is called an ARMA difference equation.

EX: $y[n] + 0.99y[n-1] + 0.98y[n-2] =$
$x[n] + x[n-1] + x[n-2]$
This removes $A\sin(\frac{2\pi}{3}n)$ from $x[n]$.

IMPULSE AND IMPULSE RESPONSE [3/4]

DEF: A discrete time *impulse* $\delta[n]$ is:

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}.$$

DEF: The *impulse response* $h[n]$ of a system is the system's response to an impulse.

That is: Impulse $\delta[n] \rightarrow \boxed{\textbf{SYSTEM}} \rightarrow h[n]$=impulse response.

Why are LTI Systems Important? [4/4]

1. If the impulse response $h[n]$ is known, then the response to **any** input $x[n]$ is $y[n] = h[n] * x[n]$ where *=*convolution*. Coming soon in this lecture.

2. $\cos(\omega n) \to \boxed{\mathbf{h[n]}} \to A\cos(\omega n + \theta)$

where: $\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = Ae^{j\theta}$.

So? Leads to *frequency response* and *filtering*. Coming in a couple of weeks.

DEF: A system is **CAUSAL** if it has this form for some function $F(\cdot)$:
$y[n] = F(x[n], x[n-1], x[n-2] \ldots)$
(present and past input only).

LTI: LTI system is causal$\Leftrightarrow$ its impulse response $h[n]$ is causal:

That is: $h[n] = 0$ for $n < 0$.

Note: Physical systems must be causal. But DSP filters need not be causal!

DEF: A system is **MEMORYLESS** if
$y[n] = F(x[n])$ (present input only).

EX: $y[n] = \sin(x[n])/(x[n]^2 + 7n)$

That is: System is Zen ("Lives in the now")

DEF: A system is **(BIBO) STABLE** if:

Let $x[n] \to \boxed{\text{SYSTEM}} \to y[n]$.
If $|x[n]| \le M$ for some constant M,
then $|y[n]| \le N$ for some N.

That is: "Every bounded input (BI) yields bounded output (BO)."

Q: How to tell if an LTI system is BIBO stable?

A: BIBO stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (finite).

That is: Impulse response is *absolutely summable*.

Proof: Coming next lecture (uses convolution).

EX: A time-invariant system is observed to have the following two responses:

1. $\{\underline{0}, 0, 3\} \to \boxed{\textbf{SYSTEM}} \to \{\underline{0}, 1, 0, 2\}$
2. $\{\underline{0}, 0, 0, 1\} \to \boxed{\textbf{SYSTEM}} \to \{1, \underline{2}, 1\}$.

Prove: The system is nonlinear.

Proof: By contradiction. *Suppose* is linear.
Then system is LTI and we have:
$\{\underline{0}, 0, 0, 1\} \to \boxed{\textbf{SYSTEM}} \to \{1, \underline{2}, 1\}$ (given)
so $\{\underline{0}, 0, 1\} \to \boxed{\textbf{SYSTEM}} \to \{1, 2, \underline{1}\}$ (TI)
and $\{\underline{0}, 0, 3\} \to \boxed{\textbf{SYSTEM}} \to \{3, 6, \underline{3}\}$ (scale)

But: So $\{\underline{0}, 0, 3\}$ produces two outputs!

SIFTING REPRESENTATION

$x[n] = \{3, \underline{1}, 4, 6\}$ is equivalent to:
$3\delta[n+1] + 1\delta[n] + 4\delta[n-1] + 6\delta[n-2]$.

Note: $x[n] = \sum_i x[i]\delta[n-i] = x[n] * \delta[n]$

Delay: $x[n-D]$ is $x[n]$ shifted:
right (later) if $D > 0$;
left (earlier) if $D < 0$.

Fold: $x[-n]$ is $x[n]$ flipped/folded/reversed around vertical axis $n = 0$.

Both: $x[N-n]$ is $x[-n]$ shifted *right* if $N > 0$ (since $x[0]$ is now at $n = N$).

CONVOLUTION DERIVATION: [1/6]

1. $\delta[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n]$ Definition of **Impulse response** $h[n]$.

CONVOLUTION DERIVATION: [2/6]

1. $\delta[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n]$ Definition of **Impulse response** $h[n]$.
2. $\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n-i]$ **Time invariant:** delay by i.

CONVOLUTION DERIVATION: [3/6]

1. $\delta[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n]$ Definition of **Impulse response** $h[n]$.
2. $\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n-i]$ **Time invariant:** delay by i.
3. $x[i]\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow x[i]h[n-i]$ **Linear:** scale by $x[i]$.

CONVOLUTION DERIVATION: [4/6]

1. $\delta[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n]$ Definition of **Impulse response** $h[n]$.
2. $\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n-i]$ **Time invariant:** delay by i.
3. $x[i]\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow x[i]h[n-i]$ **Linear:** scale by $x[i]$.
4. $\sum_i x[i]\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow \sum_i x[i]h[n-i]$ **Linear:** superposition.

CONVOLUTION DERIVATION: [5/6]

1. $\delta[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n]$ Definition of **Impulse response** $h[n]$.
2. $\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n-i]$ **Time invariant:** delay by i.
3. $x[i]\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow x[i]h[n-i]$ **Linear:** scale by $x[i]$.
4. $\sum_i x[i]\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow \sum_i x[i]h[n-i]$ **Linear:** superposition.
5. $x[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n] = \sum_i x[i]h[n-i] = h[n] * x[n]$ **Convolution**

CONVOLUTION DERIVATION: [6/6]

1. $\delta[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n]$ Definition of **Impulse response** $h[n]$.
2. $\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow h[n-i]$ **Time invariant:** delay by i.
3. $x[i]\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow x[i]h[n-i]$ **Linear:** scale by $x[i]$.
4. $\sum_i x[i]\delta[n-i] \rightarrow \boxed{\textbf{LTI}} \rightarrow \sum_i x[i]h[n-i]$ **Linear:** superposition.
5. $x[n] \rightarrow \boxed{\textbf{LTI}} \rightarrow y[n] = \sum_i x[i]h[n-i] = h[n] * x[n]$ **Convolution**

Input $x[n]$ into LTI system with no initial stored energy→output

PROPERTIES OF CONVOLUTION [1/6]

1. y[n]=h[n]*x[n]=x[n]*h[n]=$\sum_i h[i]x[n-i]$=$\sum_i h[n-i]x[i]$.

PROPERTIES OF CONVOLUTION [2/6]

1. y[n]=h[n]*x[n]=x[n]*h[n]=$\sum_i h[i]x[n-i]$=$\sum_i h[n-i]x[i]$.
2. $h[n], x[n]$ both causal ($h[n]=0$ and $x[n]=0$ for $n<0$)
   $\rightarrow y[n]$=$\sum_{i=0}^{n} h[i]x[n-i]$=$\sum_{i=0}^{n} h[n-i]x[i]$ also causal.

PROPERTIES OF CONVOLUTION [3/6]

1. y[n]=h[n]*x[n]=x[n]*h[n]=$\sum_i h[i]x[n-i]$=$\sum_i h[n-i]x[i]$.
2. $h[n], x[n]$ both causal ($h[n]=0$ and $x[n]=0$ for $n<0$)
   $\rightarrow y[n]$=$\sum_{i=0}^{n} h[i]x[n-i]$=$\sum_{i=0}^{n} h[n-i]x[i]$ also causal.
3. Let $h[n] \neq 0$ *only for* 0≤n≤L ($h[n]$ has length $L+1$).
   Let $x[n] \neq 0$ *only for* 0≤n≤M ($x[n]$ has length $M+1$).
   Then $y[n] \neq 0$ only for 0≤n≤L+M ($y[n]$ has length $L+M+1$).

Note: Length[$y[n]$]=Length[$h[n]$]+Length[$x[n]$]–1

Note: y[0]=h[0]x[0]; y[L+M]=h[L]x[M]; x[n]*$\delta[n-D]$=x[n-D].

EXAMPLE OF THESE PROPERTIES [4/6]

$y[n] = \{\underline{1}, 2, 3\} * \{\underline{4}, 5, 6\} =?$

$y[n] = 0$ for $n < 0$ (all causal)
Length($y[n]$)=3+3–1=5.
$y[0] = h[0]x[0] = (1)(4) = 4.$
$y[4] = h[2]x[2] = (3)(6) = 18.$
$y[n] = \{\underline{4}, ?, ?, ?, 18\}.$

PROPERTIES OF CONVOLUTION [5/6]

4. $x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow y[n]$ (cascade connection)
   Equivalent to: x[n]$\rightarrow \boxed{h_1[n] * h_2[n]} \rightarrow y[n]$. (Associative property)

Rewrite: $x[n] \rightarrow \boxed{h_1[n]} \rightarrow w[n] \rightarrow \boxed{h_2[n]} \rightarrow y[n]$

Then: w[n]=$h_1[n]$*x[n] and y[n]=$h_2[n]$*w[n]

and: y[n]=$h_2[n]$*w[n]=$h_2[n]$*($h_1[n]$*x[n])=($h_2[n] * h_1[n]$)*x[n].

PROPERTIES OF CONVOLUTION [6/6]

4. $x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow y[n]$ (cascade connection)
   Equivalent to: x[n]$\rightarrow \boxed{h_1[n] * h_2[n]} \rightarrow y[n]$. (Associative property)
5. $x[n] \rightarrow$ [$\rightarrow \boxed{h_1[n]} \rightarrow$ / $\rightarrow \boxed{h_2[n]} \rightarrow$] $\bigoplus \rightarrow y[n]$ (parallel connection)
   Equivalent to: $x[n] \rightarrow \boxed{h_1[n] + h_2[n]} \rightarrow y[n]$. (Distributive proper