TOPICS FOR TODAY'S LECTURE

1. 2-sided Inverse z-transform
   a. Sum of geometrics
   b. ROC→causal or anticausal
2. ROCs: Causality vs. Stability

NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [1/2]

Given: $X(z) = A_1 \frac{z}{z-p_1} + \ldots + A_N \frac{z}{z-p_N}$.

Goal: Compute all possible $\mathcal{Z}^{-1}$ of $X(z)$.

Soln: $x[n] = \begin{cases} A_1 p_1^n u[n] & \text{if ROC} \subset \{|z| > |p_1|\} \\ -A_1 p_1^n u[-n-1] & \text{if ROC} \subset \{|z| < |p_1|\} \end{cases}$

$\ldots + \begin{cases} A_N p_N^n u[n] & \text{if ROC} \subset \{|z| > |p_N|\} \\ -A_N p_N^n u[-n-1] & \text{if ROC} \subset \{|z| < |p_N|\} \end{cases}$

NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [2/2]

ROC: $\bigcap_{n=1}^{N} \{|z| > |p_n| \text{ OR } |z| < |p_n|\}$.

But: $N+1$ possible $\mathcal{Z}^{-1}$, not 2^N.

Recall: Maximum (magnitude) causal pole< minimum (magnitude) anticausal pole

EX #1: NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [1/3]

Goal: Compute all possible $\mathcal{Z}^{-1}$ of:

$X(z) = \frac{2z^2 - 2.5z}{z^2 - 2.5z + 1}$.

Partial Fraction: $\frac{X(z)}{z} = \frac{2z-2.5}{(z-2)(z-\frac{1}{2})} = \frac{1}{z-\frac{1}{2}} + \frac{1}{z-2}$.

Mult. by z: $X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-2}$. Now need $\mathcal{Z}^{-1}$.

EX #1: NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [2/3]

The 3 inverse z-transforms are:

INVERSE zXFORM	ROC	COMMENT
$(\frac{1}{2})^n u[n] + (2)^n u[n]$	$\|z\| > 2$	causal
$(\frac{1}{2})^n u[n] - (2)^n u[-n-1]$	$\frac{1}{2} < \|z\| < 2$	stable
$-(\frac{1}{2})^n u[-n-1] - (2)^n u[-n-1]$	$\|z\| < \frac{1}{2}$	anticausal

EX #1: NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [3/3]

But: This is NOT a valid inverse z-transform: x[n]=$-(\frac{1}{2})^n u[-n-1] + (2)^n u[n]$! Since:

ROC: $\{|z| < \frac{1}{2}\} \cap \{|z| > 2\} = \emptyset$.

So: $\mathcal{Z}\{x[n]\} = X(z)$ for NO values of z!

EX #2: NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [1/3]

Goal: Compute all inverse z-transforms of $\frac{z}{z-2} + \frac{z}{z-3} + \frac{z}{z-4}$.

Sol'n: $\begin{cases} (2)^n u[n] \\ -(2)^n u[-n-1] \end{cases} + \begin{cases} (3)^n u[n] \\ -(3)^n u[-n-1] \end{cases} + \begin{cases} (4)^n u[n] \\ -(4)^n u[-n-1] \end{cases}$

Seems: $2^3 = 8$ possible inverse z-transforms.
But: Look at the ROCs: which are not $\emptyset$?

EX #2: NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [2/3]

1. $|z| < 2 \cap |z| < 3 \cap |z| < 4 = |z| < 2$.
2. $|z| > 2 \cap |z| < 3 \cap |z| < 4 = 2 < |z| < 3$.
3. $|z| < 2 \cap |z| > 3 \cap |z| < 4 = \emptyset$.
4. $|z| > 2 \cap |z| > 3 \cap |z| < 4 = 3 < |z| < 4$.
5. $|z| < 2 \cap |z| < 3 \cap |z| > 4 = \emptyset$.
6. $|z| > 2 \cap |z| < 3 \cap |z| > 4 = \emptyset$.
7. $|z| < 2 \cap |z| > 3 \cap |z| > 4 = \emptyset$
8. $|z| > 2 \cap |z| > 3 \cap |z| > 4 = |z| > 4$.

EX #2: NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [3/3]

So: 3+1=4 possible inverse z-transforms are:

$|z| < 2$: $-(2)^n$u[-n-1]$-(3)^n$u[-n-1]$-(4)^n$u[-n-1] (anticausal)

$2 < |z| < 3$: $(2)^n$u[n]$-(3)^n$u[-n-1]$-(4)^n$u[-n-1] (two-sided)

$3 < |z| < 4$: $(2)^n$u[n]$+(3)^n$u[n]$-(4)^n$u[-n-1] (two-sided)

$|z| > 4$: $(2)^n$u[n]$+(3)^n$u[n]$+(4)^n$u[n] (causal)

EX #3: NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [1/4]

Goal: $X(z) = \frac{120}{(z-1)(z-2)(z-3)(z-4)(z-5)}$.

ROC: $3 < |z| < 4$. Compute $\mathcal{Z}^{-1}$.

Note: $x[n]$ will be 2-sided.

EX #3: NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [2/4]

Residue: A_0=(z–0)X(z)/z$|_{z=0}$=120/[(0–1)(0–2)(0–3)(0–4)(0–5)]=–1.
Residue: A_1=(z–1)X(z)/z$|_{z=1}$=120/[(1–0)(1–2)(1–3)(1–4)(1–5)]=5.
Residue: A_2=(z–2)X(z)/z$|_{z=2}$=120/[(2–0)(2–1)(2–3)(2–4)(2–5)]=–10.
Residue: A_3=(z–3)X(z)/z$|_{z=3}$=120/[(3–0)(3–1)(3–2)(3–4)(3–5)]=10.
Residue: A_4=(z–4)X(z)/z$|_{z=4}$=120/[(4–0)(4–1)(4–2)(4–3)(4–5)]=–5.
Residue: A_5=(z–5)X(z)/z$|_{z=5}$=120/[(5–0)(5–1)(5–2)(5–3)(5–4)]=1.

EX #3: NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [3/4]

Compute: Partial fraction expansion of $X(z)/z$.
Recall: $X(z) = 120/[(z-1)(z-2)(z-3)(z-4)(z-5)]$

Then: $\frac{X(z)}{z} = \frac{-1}{z} + \frac{5}{z-1} - \frac{10}{z-2} + \frac{10}{z-3} - \frac{5}{z-4} + \frac{1}{z-5}$.

$\mathcal{Z}^{-1}: \; x[n] = -\delta[n] + 5\begin{cases} u[n] \\ -u[-n-1] \end{cases} - 10\begin{cases} 2^n u[n] \\ -2^n u[-n-1] \end{cases} + \cdots$

EX #3: NON-CAUSAL INVERSE 2-SIDED Z-TRANSFORMS [4/4]

$\mathcal{Z}^{-1}$: $-\delta[n]+5u[n]-10(2)^n u[n]+10(3)^n u[n]+5(4)^n u[-n-1]-5^n u[-n-1]$

since: ROC$= \{3 < |z| < 4\} \subset \{|z| > 1, |z| > 2, |z| > 3, |z| < 4, |z| < 5\}$.

Note: 6 possible ROCs:
$\{|z| < 1\}; \{1 < |z| < 2\}; \{2 < |z| < 3\} \ldots \{4 < |z| < 5\}; \{5 < |z|\}$.

STABILITY AND CAUSALITY [1/5]

1. Stable$\Leftrightarrow \{|z| = 1\} \subset$ ROC.
2. Causal$\Leftrightarrow$ ROC $= \{|z| > A\}$
3. $\substack{\text{Causal}\\\text{Stable}} \Leftrightarrow$ ROC $= \{|z| > A < 1\}$
4. $\substack{\text{Causal}\\\text{Stable}} \rightarrow |p_i| < 1$ (poles inside unit circle).
5. $\substack{\text{Anticausal}\\\text{and Stable}} \rightarrow |p_i| > 1$ (poles outside unit circle).
6. Poles inside unit circle$\rightarrow \substack{\text{Causal}\\\text{Stable}}$ OR $\substack{\text{Anticausal}\\\text{Unstable}}$.

IVT: $x(n)$ causal$\rightarrow \lim_{z\to\infty} X(z) = x[0]$
since $X(z) = x[0] + x[1]z^{-1} + \ldots$

STABILITY AND CAUSALITY [2/5]

Given: $\mathcal{Z}\{x[n]\} = X(z)$ & ROC.
But: We don't know $X(z)$, only ROC!
Then: What can we say about $x[n]$?

1. ROC$=\{z : |z| > a > 1\} \rightarrow$?
2. ROC$=\{z : |z| > a < 1\} \rightarrow$?
3. ROC$=\{z : |z| < a < 1\} \rightarrow$?
4. ROC$=\{z : |z| < a > 1\} \rightarrow$?

STABILITY AND CAUSALITY [3/5]

1. ROC$=\{z : |z| > a > 1\}$ $\rightarrow$causal & unstable
2. ROC$=\{z : |z| > a < 1\}$ $\rightarrow$causal & stable
3. ROC$=\{z : |z| < a < 1\}$ $\rightarrow$anticausal & unstable
4. ROC$=\{z : |z| < a > 1\}$ $\rightarrow$anticausal & stable

STABILITY AND CAUSALITY [4/5]

1. ROC$=\{z : 1 < a < |z| < b\} \rightarrow$?
2. ROC$=\{z : a < |z| < b < 1\} \rightarrow$?
3. ROC$=\{z : 1 > a < |z| < b > 1\} \rightarrow$?
4. ROC$=\{z : 1 < a < |z| < b < 1\} \rightarrow$?

STABILITY AND CAUSALITY [5/5]

1. ROC$=\{z : 1 < a < |z| < b\}$ $\rightarrow$2-sided & unstable as $n \rightarrow \infty$
2. ROC$=\{z : a < |z| < b < 1\}$ $\rightarrow$2-sided & unstable as $n \rightarrow -\infty$
3. ROC$=\{z : 1 > a < |z| < b > 1\}$ $\rightarrow$2-sided & stable
4. ROC$=\{z : 1 < a < |z| < b < 1\}$ $\rightarrow$Impossible!