TOPICS FOR TODAY'S LECTURE

CAUSAL INVERSE z-TRANSFORMS

1. Simple: Do by inspection
2. Rational functions (ratio of polynomials):
   a. All real and distinct poles
   b. Complex and distinct poles
   c. Multiple poles at origin
3. Use of Matlab's `residue` for these

Next time: 2-sided inverse z-xform

INVERSES BY INSPECTION [1/2]

EX #1: $\mathcal{Z}^{-1}\{2z+3+4z^{-2}\}$. **Soln:** $\{2,\underline{3},0,4\}$.

EX #2: $\mathcal{Z}^{-1}\{\frac{2z-3}{z}\}$. **Soln:** $\frac{2z-3}{z}=2-3z^{-1}\rightarrow\{\underline{2},-3\}$.

Or: $\mathcal{Z}^{-1}\{2z-3\}=\{2,\underline{-3}\}\rightarrow\mathcal{Z}^{-1}\{\frac{2z-3}{z}\}=\{\underline{2},-3\}$.

Recall: Multiply $X(z)$ by $z^{-1}\rightarrow$ Replace n by n–1 **everywhere**.

INVERSES BY INSPECTION [2/2]

EX #3: $\mathcal{Z}^{-1}\{2\frac{z-1}{z-2}\}$. The following both=$\{\underline{2},2,4,8,16\ldots\}$:

Soln: $X(z)=\frac{2z}{z-2}-z^{-1}\frac{2z}{z-2}\rightarrow x[n]=2(2)^n u[n]-2(2)^{n-1}\text{u}[\text{n}-1]$.

Or: $X(z)=2\frac{z-2}{z-2}+2\frac{1}{z-2}\rightarrow x[n]=2\delta[n]+2(2)^{n-1}\text{u}[\text{n}-1]$.

RATIONAL FUNCTIONS

Given: $X(z)=\frac{b_0+b_1z+\ldots+b_Mz^M}{a_0+a_1z+\ldots+a_Nz^N}$. **Assume:** $M\leq N$.

Divide z: $\frac{X(z)}{z}=\frac{b_0+b_1z+\ldots+b_Mz^M}{a_0z+a_1z^2+\ldots+a_Nz^{N+1}}$ **Note:** $M<N+1$.

$=$ RATIO OF TWO POLYNOMIALS $=$ RATIONAL FUNCTION. Use PARTIAL FRACTION.

POLES OF RATIONAL FUNCTION

Poles: $\{0,p_1\ldots p_N\}$ are roots of $a_0z+\ldots+a_Nz^{N+1}=0$.
Assume: The poles $\{0,p_n\}$ distinct (as happens in real life).

That is: Poles are the roots of denominator=0.
Compute: Poles in Matlab: `roots(`$[a_N,\ldots a_0,0]$`)`
Note: a_n real→poles in complex conjugate pairs.

PARTIAL FRACTIONS

First: Divide $X(z)$ by z, then compute its Partial Fraction Expansion (PFE):
PFE: $\frac{X(z)}{z}=\frac{A_0}{z}+\frac{A_1}{z-p_1}+\ldots+\frac{A_N}{z-p_N}$
Since: Distinct poles: $0\neq p_1\neq\ldots\neq p_N$.

Residues: $A_n=(z-p_n)\frac{X(z)}{z}$ at $z=p_n$
After: Cancelling the $(z–p_n)$ terms.
Complex conjugate: $p_{n+1}=p_n^*\rightarrow A_{n+1}=A_n^*$.

CAUSAL INVERSE z-TRANSFORM

PFE: $\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-p_1} + \ldots + \frac{A_N}{z-p_N}$

Multiply partial fraction expansion by z:

Get: $X(z) = A_0 + A_1\frac{z}{z-p_1} + \ldots + A_N\frac{z}{z-p_N}$.

$\mathcal{Z}^{-1}$: $x[n] = A_0\delta[n] + A_1 p_1^n u[n] + \ldots + A_N p_N^n u[n]$.

Is Why: Divided by z, to use $\mathcal{Z}^{-1}\{\frac{A_i z}{z-p_i}\}=A_i p_i^n u[n]$.

EXAMPLE #1 [1/2]

Goal: Compute $\mathcal{Z}^{-1}\{\frac{z-3}{z^2-3z+2}\}$.

Poles: $z^2-3z+2=0\rightarrow z=1$ or 2.

PFE: $\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_2}{z-2}$

since: $z(z^2-3z+2)=z(z-1)(z-2)$.

EXAMPLE #1 [2/2]

Residue: $A_0 = \frac{(0-3)}{(0-1)(0-2)} = -\frac{3}{2}$.

Residue: $A_1 = \frac{(1-3)}{(1-0)(1-2)} = 2$.

Residue: $A_2 = \frac{(2-3)}{(2-0)(2-1)} = -\frac{1}{2}$.

Then: $X(z) = -\frac{3}{2} + 2\frac{z}{z-1} - \frac{1}{2}\frac{z}{z-2}$.

$\mathcal{Z}^{-1}$: $x[n] = -\frac{3}{2}\delta[n] + 2u[n] - \frac{1}{2}(2)^n u[n]$.

SIMPLIFY COMPLEX EXPONENTIALS

Useful: $Ap^n + A^*(p^*)^n = 2|A||p|^n\cos(\omega_0 n + \theta)$

where: $A = |A|e^{j\theta}$ and $p = |p|e^{j\omega_0}$ (polar forms)

since: $Ap^n+A^*(p^*)^n=|A||p|^n(e^{j\theta}e^{j\omega_0 n} + e^{-j\theta}e^{-j\omega_0 n})$

EX: $(3+j4)(1+j)^n+(3-j4)(1-j)^n$

= $2(5)(\sqrt{2})^n\cos(\frac{\pi}{4}n + 53^o)$

where: $3+j4=5e^{j53^o}$ and $1+j=\sqrt{2}e^{j\pi/4}$.

Much easier than using sines and cosines directly!

EXAMPLE #2 [1/2]

Goal: Compute $\mathcal{Z}^{-1}\{\frac{2z}{z^2-2z+2}\}$.

Poles: $z^2-2z+2=0\rightarrow z=1\pm j$.

PFE: $\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-(1+j)} + \frac{A_1^*}{z-(1-j)}$

since: $z(z^2-2z+2)=z(z-(1+j))(z-(1-j))$.

EXAMPLE #2 [2/2]

Residue: $A_0 = \frac{2(0)}{(0-(1+j))(0-(1-j))} = 0$.

Residue: $A_1 = \frac{2(1+j)}{(1+j)((1+j)-(1-j))} = -j$

Then: $\frac{X(z)}{z} = \frac{-j}{z-(1+j)} + \frac{j}{z-(1-j)}$.

$\mathcal{Z}^{-1}$: $x[n] = -j(1+j)^n u[n] + j(1-j)^n u[n]$

Simplify: $x[n] = 2(\sqrt{2})^n\cos(\frac{\pi}{4}n - \frac{\pi}{2})u[n]$

since: $1+j=\sqrt{2}e^{j\pi/4}$ and $-j=e^{-j\pi/2}$.

MULTIPLE POLES AT ORIGIN

Goal: $\mathcal{Z}^{-1}\{\frac{z^3+2z^2+3z+4}{z^2(z-1)}\}$ (2 poles at origin z=0).

Soln: $X(z)=\frac{z^3+2z^2+3z+4}{z^3}\frac{z}{z-1} = (1+2z^{-1}+3z^{-2}+4z^{-3})\frac{z}{z-1}$

$\rightarrow$ x[n]=$\{\underline{1},2,3,4\}$∗u[n]=u[n]+2u[n–1]+3u[n–2]+4u[n–3]

$\rightarrow$ x[n]=$\{\underline{1},3,6,10,10,10\ldots\}$

COMPLEX COMPLEX PROBLEM [1/4]

Goal: $\mathcal{Z}^{-1}\{\frac{z-1}{z^3+4z^2+8z+8}\}$ (Chen p. 257).

Poles: $z^3+4z^2+8z+8=(z+2)(z-2e^{j2.09})(z-2e^{-j2.09})$

PFE: $\frac{X(z)}{z} = \frac{z-1}{z(z+2)(z-2e^{j2.09})(z-2e^{-j2.09})}$

$=\frac{A}{z}+\frac{B}{z+2}+\frac{C}{z-2e^{j2.09}}+\frac{C^*}{z-2e^{-j2.09}}.$

COMPLEX COMPLEX PROBLEM [2/4]

Residue: $A=(z-0)\frac{X(z)}{z}|_{z=0}=\frac{0-1}{(0+2)(0-2e^{j2.09})(0-2e^{-j2.09})}=-\frac{1}{8}$

Residue: $B=(z+2)\frac{X(z)}{z}|_{z=-2}=\frac{-2-1}{(-2)(-2-2e^{j2.09})(-2-2e^{-j2.09})}=\frac{3}{8}$

Residue: $C=(z-2e^{j2.09})\frac{X(z)}{z}|_{z=2e^{j2.09}}=\frac{(2e^{j2.09}-1)/2e^{j2.09}}{(2e^{j2.09}+2)(2e^{j2.09}-2e^{-j2.09})}$

$=0.19e^{-j2.29}$ **Recall:** Add and subtract in rect. form.

COMPLEX COMPLEX PROBLEM [3/4]

$X(z)=-\frac{1}{8}+\frac{3}{8}\frac{z}{z+2}+0.19e^{-j2.29}\frac{z}{z-2e^{j2.09}}+0.19e^{j2.29}\frac{z}{z-2e^{-j2.09}}.$

$\mathcal{Z}^{-1}$: $x[n]=-\frac{1}{8}\delta[n]+\frac{3}{8}(-2)^n u[n]+(0.19)2^n e^{j(2.09n-2.29)}u[n]$

$+(0.19)2^n e^{-j(2.09n-2.29)}u[n]$

Using: $Ap^n+A^*(p^*)^n=2|A||p|^n\cos(\omega_0 n+\theta)$, $A=|A|e^{j\theta}; p=|p|e^{j\omega_0}$,

Simplify: $x(n)=-\frac{1}{8}\delta[n]+\frac{3}{8}(-2)^n u[n]+(0.38)2^n\cos(2.09n-2.29)u[n]$.

COMPLEX COMPLEX PROBLEM [4/4]

Matlab: `>> [R P]=residue([1 -1],[1 4 8 8 0]);`
Then: `>> [abs(P) angle(P) abs(R) angle(R)]`
Gives: `2.0000 2.0944 0.1909 -2.2845`
(Actual `2.0000 -2.0944 0.1909 2.2845`
output) `2.0000 3.1416 0.3750 0.0000`
`0.0000 0.0000 0.1250 3.1416`

Get: $-\frac{1}{8}\delta[n]+\frac{3}{8}(-2)^n u[n]+(0.3818)2^n\cos(2.0944n-2.2845)u[n]$.
Note: 2(0.1909)=0.3818, $0.125e^{j3.1416}=-\frac{1}{8}$, etc. Same as above.