TOPICS FOR TODAY'S LECTURE

FAST FOURIER TRANSFORM (FFT)

1. Derivation using $\mathcal{Z}$ xform
2. Derivation: General case
3. Radix-2 FFT (N=2^n)
   a. Decimation-in-time
   b. Decimation-in-freq.
4. Small radix-2 example

INTRODUCTION [1/2]

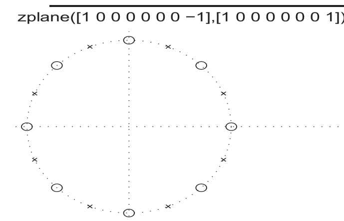
Goal: Compute DFT $X_k = \sum_{n=0}^{N-1} x[n] W_N^{nk}$
where: $W_N = e^{-j2\pi/N}$ (makes formulae neater).
FFT: *Algorithm* for computing:
DFT: A *transform* $x[n] \rightarrow X_k$.
Direct: N^2 Mults. and Adds (MADs).
History: Gauss; Cooley-Tukey (1965).
Note: DSP dates from this algorithm.

INTRODUCTION [2/2]

Idea: *Divide* large problem into smaller;
Then: *Conquer* by solving smaller ones.
EX: 1000-point problem: $(1000)^2$=1 million ops.
But: 100 10-point problems: $100(10)^2$=10,000 ops.
But: Great! But how can we do this?

z-TRANSFORM DERIVATION [1/2]

Recall: $X_k = X(z)|_{z=e^{j2\pi k/N}}$.
That is: Evaluate $X(z)$ at:



z-TRANSFORM DERIVATION [2/2]

Idea: $X_{2k} = X(z)|_{z=e^{j2\pi 2k/N}} = X(z)|_{z=e^{j2\pi k/(N/2)}}$.
and: $X_{2k+1} = X(z)|_{z=e^{j2\pi(2k+1)/N}} = X(z)|_{z=e^{j2\pi/N}e^{j2\pi k/(N/2)}}$.
That is: Even indices: $\frac{N}{2}$-point DFT of x[n]+x[n+$\frac{N}{2}$] (aliased)
and: Odd indices: $\frac{N}{2}$-point DFT of $e^{j2\pi n/N}$(x[n]–x[n+$\frac{N}{2}$]) (modulate)
Why: *Aliased*: see HW. The −*sign*: $e^{j2\pi(n+(N/2)/N} = -e^{j2\pi n/N}$.

GENERAL DERIVATION [1/5]

Goal: (N_1N_2)-point DFT→
N_2 (N_1-point) & N_1 (N_2-point).
DEF: Coarse and vernier indices:
Coarse: $n = n_1 + N_1 n_2$ where ${}^{n_1=0\ldots N_1-1}_{n_2=0\ldots N_2-1}$.
Vernier: $k = k_2 + N_2 k_1$ where ${}^{k_1=0\ldots N_1-1}_{k_2=0\ldots N_2-1}$.
Why: Coarse and fine microscope controls.

GENERAL DERIVATION [2/5]

nk$=(n_1+N_1n_2)(k_2+N_2k_1)$
$=n_1k_2+N_1n_2k_2+N_2n_1k_1+N_1N_2n_2k_1$.

$e^{-j2\pi(\cdot)}$: $W_N^{nk}=W_N^{n_1k_2}W_{N_2}^{n_2k_2}W_{N_1}^{n_1k_1}$

using: $W_{N_1N_2}^{N_1}=W_{N_2}$ and $W_N^{N_1N_2}=1$.

GENERAL DERIVATION [3/5]

DFT: $X_k=X_{k_2+N_2k_1}=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}x[n_1+N_1n_2]W_N^{(n_1+N_1n_2)(k_2+N_2k_1)}$

Rewrite: $$X_k=X_{k_2+N_2k_1}=\underbrace{\sum_{n_1=0}^{N_1-1}W_{N_1}^{n_1k_1}}_{\mathrm{N_1-point\ DFT}}[W_N^{n_1k_2}\underbrace{\sum_{n_2=0}^{N_2-1}W_{N_2}^{n_2k_2}x[n_1+N_1n_2]]}_{\mathrm{N_2-point\ DFT\ for\ each\ n_1}}.$$

GENERAL DERIVATION [4/5]

1. Compute N_1 (for each n_1) N_2-point DFTs of $x[n_1+N_1n_2]$.
2. Multiply result by *twiddle factors* $W_N^{n_1k_2}$. Called *twiddle mults*.
3. Compute N_2 (for each k_2) N_1-point DFTs of the result. Done!

(N_1N_2) –point→ N_1(N_2-point)+N_2(N_1-point)+$(N_1–1)(N_2–1)$ twiddle.
since: $n_1=0$ or $k_2=0\rightarrow W_N^{n_1k_2}=1$. This is important below.

GENERAL DERIVATION [5/5]

Visual: $(N_1\times N_2)$ arrays: x_{n_1,n_2}=x$[n_1+N_1n_2]$; $X_{k_1,k_2}=X_{k_2+N_2k_1}$.

1. Take N_2-point DFT of each row (fixed n_1). Yields $\hat{x}_{n_1,k_2}$.
2. Multiply $\hat{x}_{n_1,k_2}$ point-by-point by the twiddle factor $W_N^{n_1k_2}$.
3. Take N_1-point DFT of each column (fixed k_2). Yields X_{k_1,k_2}.

Example: Later in lecture.

RADIX-2 COOLEY-TUKEY FFT

1. *Decimation-in-time:* $N=2\frac{N}{2}$.

Set: $N_1=2;N_2=N/2;\quad n_1=0,1;k_1=0,1$.

2. *Decimation-in-freq:* $N=\frac{N}{2}2$.

Set: $N_1=N/2;N_2=2;\quad n_2=0,1;k_2=0,1$.

Then: N-point→2($\frac{N}{2}$-point)+$\frac{N}{2}$(2-point)
+($\frac{N}{2}$–1) twiddle mults. **Total:** $\frac{N}{2}\log_2 N$ mults.

Note: $(N_1–1)(N_2–1)=(2–1)(\frac{N}{2}–1)=\frac{N}{2}–1$:
That is: Half the twiddle mults are trivial!

DEC.-IN-TIME RADIX-2 COOLEY-TUKEY FFT

$$X_{k_2}=1\underbrace{\sum_{n_2=0}^{N/2-1}W_{N/2}^{n_2k_2}x[2n_2]}_{\mathrm{N/2-point\ DFT}}+W_N^{1k_2}\underbrace{\sum_{n_2=0}^{N/2-1}W_{N/2}^{n_2k_2}x[2n_2+1]}_{\mathrm{N/2-point\ DFT}}.$$

$$X_{k_2+N/2}=1\underbrace{\sum_{n_2=0}^{N/2-1}W_{N/2}^{n_2k_2}x[2n_2]}_{\mathrm{N/2-point\ DFT}}-W_N^{1k_2}\underbrace{\sum_{n_2=0}^{N/2-1}W_{N/2}^{n_2k_2}x[2n_2+1]}_{\mathrm{N/2-pointDFT}}.$$

DEC.-IN-FREQ RADIX-2 COOLEY-TUKEY FFT

$$X_{2k_1} = \underbrace{\sum_{n_1=0}^{N/2-1} W_{N/2}^{n_1 k_1}}_{\text{N/2−point DFT}} \underbrace{[x[n_1] + x[n_1+N/2]]}_{\text{Half of 2−point DFT}} 1.$$

$$X_{2k_1+1} = \underbrace{\sum_{n_1=0}^{N/2-1} W_{N/2}^{n_1 k_1}}_{\text{N/2−point DFT}} \underbrace{[x[n_1] - x[n_1+N/2]]}_{\text{Half of 2−point DFT}} W_N^{1n_1}.$$

1. $\boxed{\begin{matrix} x[0] & x[1] \\ x[2] & x[3] \end{matrix}} = \boxed{\begin{matrix} a & b \\ c & d \end{matrix}} = \boxed{\begin{matrix} a \\ c \end{matrix}} \boxed{\begin{matrix} b \\ d \end{matrix}} \rightarrow \boxed{\begin{matrix} (a+c) \\ (a-c) \end{matrix}} \boxed{\begin{matrix} (b+d) \\ (b-d) \end{matrix}}$

2. $\boxed{\begin{matrix} (a+c) & (b+d) \\ (a-c) & (b-d) \end{matrix}} \times \boxed{\begin{matrix} 1 & 1 \\ 1 & -j \end{matrix}} = \boxed{\begin{matrix} (a+c) & (b+d) \\ (a-c) & -j(b-d) \end{matrix}}$

3. $\boxed{[(a+c)+(b+d)] \quad [(a+c)-(b+d)]} = \boxed{X_0 \quad X_2}$

$\boxed{[(a-c)-j(b-d)] \quad [(a-c)+j(b-d)]} = \boxed{X_1 \quad X_3}$

4. Read off $\{X_0, X_1, X_2, X_3\}$. Note *index shuffling.*

COMPUTING $\mathcal{F}$ USING DFT [1/3]

Goal: Compute *numerically*

$\mathcal{F}$: X(f)=$\int x(t)e^{-j2\pi ft}dt$.

$\mathcal{F}^{-1}$: x(t)=$\int X(f)e^{j2\pi ft}df$.

Note: X(f)=$X(\frac{\omega}{2\pi})$ & df=$\frac{d\omega}{2\pi}$.

Assume: $x(t)$ *time-limited* to $0 < t < T$.

Assume: $X(f)$ *band-limited* to $|f| < B/2$.

COMPUTING $\mathcal{F}$ USING DFT [2/3]

Sample t: $t = n\Delta_t$ for $0 \leq n \leq N-1$.
Sample f: $f = k\Delta_f$ for $0 \leq k \leq N-1$.

Nyquist: Δ_t=1/B and Δ_f=1/T.

Approx: $X(f) \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi fn\Delta_t}\Delta_t$.

Sample f: $X(k\Delta_f) \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi nk\Delta_t\Delta_f}\Delta_t$.

and: $x(n\Delta_t) \approx \sum_{k=0}^{N-1} X(k\Delta_f)e^{j2\pi nk\Delta_t\Delta_f}\Delta_f$.

COMPUTING $\mathcal{F}$ USING DFT [3/3]

Let: X_k=$X(k\Delta_f)\frac{1}{\Delta_t}$=$X(k\Delta_f)B$
and: $\Delta_t\Delta_f$=$\frac{1}{N}$ and N=BT.

Then: $X_k \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi nk/N}$.

and: $x(n\Delta_t) \approx \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi nk/N}$.

Formulae: $BT = N$; $\Delta_t\Delta_f = 1/N$ $x[n] = X(n\Delta_t)$.
Formulae: Δ_t=$1/B$; $\Delta_f = 1/T$; $X_k = X(k\Delta_f)B$.

COMPUTING $\mathcal{F}$ USING DFT: EXAMPLE

EX: $\mathcal{F}\{e^{-|t|}\}$=$\frac{2}{\omega^2+1}$=$\frac{2}{4\pi^2 f^2+1}$.

Limits: $-6 < t < 6 \rightarrow T = 12$
and: $-8 < f < 8 \rightarrow B = 16$.

Values: N=BT=(16)(12)=192;
and: $\Delta_t = \frac{1}{B} = \frac{1}{16}$; $\Delta_f = \frac{1}{T} = \frac{1}{12}$.

FFT: 192-point FFT of $x(t)$ sampled t=n/16
gives: $16X(f)$ sampled at f=k/12.

Result: Actual & computed coincide.