TOPICS FOR TODAY'S LECTURE

FAST FOURIER TRANSFORM (FFT)
1. Derivation using $Z$-transform
2. Derivation: General case
3. Radix-2 FFT ($N=2^n$)
   a. Decimation-in-time
   b. Decimation-in-frequency
4. Small radix-2 example

INTRODUCTION [1/2]

Goal: Compute DFT $X_k = \sum_{n=0}^{N-1} x[n]W_n^k$ where: $W_N = e^{-j2\pi/N}$ (makes formulae neater).
FFT: Algorithm for computing:
DFT: A transform $x[n] \rightarrow X_k$.
Direct: $N^2$ Mults. and Adds (MADs).
History: Gauss; Cooley-Tukey (1965).
Note: DSP dates from this algorithm.

INTRODUCTION [2/2]

Idea: Divide large problem into smaller;
Then: Conquer by solving smaller ones.
EX: 1000-point problem: $(1000)^2=1$ million ops.
But: 100 10-point problems: $100(10)^2=10,000$ ops.
But: Great! But how can we do this?

z-TRANSFORM DERIVATION [1/2]

Recall: $X_k = X(z)|_{z=e^{j2\pi k/N}}$.
That is: Evaluate $X(z)$ at:
$zplane(\{1 0 0 0 0 0 0 −1\} \{1 0 0 0 0 0 0 1\})$

z-TRANSFORM DERIVATION [2/2]

Idea: $X_{2k} = X(z)|_{z=e^{j2\pi 2k/N}} = X(z)|_{z=e^{j2\pi k/(N/2)}}$.
and: $X_{2k+1} = X(z)|_{z=e^{j2\pi (2k+1)/N}} = X(z)|_{z=e^{j2\pi k/(N/2)}}$.
That is: Even indices: $N/2$-point DFT of $x[n]+x[n+N/2]$ (aliased)
and: Odd indices: $N/2$-point DFT of $e^{j2\pi n/N}(x[n]+x[n+N/2])$ (modulate)
Why: Aliased: see HW. The $−$ sign: $e^{j2\pi (n+(N/2))/N} = −e^{j2\pi n/N}$.

GENERAL DERIVATION [1/5]

Goal: ($N_1 N_2$)-point DFT $\rightarrow$
$N_2$ ($N_1$-point) & $N_1$ ($N_2$-point).
DEF: Coarse and vernier indices:
Coarse: $n = n_1 + N_1 n_2$ where $n_1=0...N_1-1$ $n_2=0...N_2-1$.
Vernier: $k = k_2 + N_2 k_1$ where $k_1=0...N_1-1$ $k_2=0...N_2-1$.
Why: Coarse and fine microscope controls.
GENERAL DERIVATION [2/5]

\[ nk = (n_1 + N_1 n_2)(k_2 + N_2 k_1) = n_1 k_2 + N_1 n_2 k_1 + N_2 n_1 k_1 + N_1 N_2 n_1 n_2 k_1. \]

\[ e^{-j2\pi n k}W^N_N = W^{n_1 k_2}W^{n_2 k_1}W^{n_1 k_1} \]

using: \( W^{N_1}_{N_1N_2} = W_{N_2} \) and \( W_N^{N_1N_2} = 1. \)

GENERAL DERIVATION [3/5]

**DFT:** \( X_k = X_{k_2 + N_2 k_1} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1 + N_1 n_2]W_N^{n_1 + N_1 n_2}(k_2 + N_2 k_1); \)

**Rewrite:** \( X_k = X_{k_2 + N_2 k_1} = \sum_{n_1=0}^{N_1-1} W_{N_1}^{n_1 k_1} \left[ W_{N_2}^{n_2 k_2} \sum_{n_2=0}^{N_2-1} W_N^{n_1 k_2} x[n_1 + N_1 n_2] \right]. \)

\[ \text{N}_1 \text{-point DFT} \quad \text{N}_2 \text{-point DFT for each } n_1 \]

GENERAL DERIVATION [4/5]

1. Compute \( N_1 \) (for each \( n_1 \)) \( N_2 \)-point DFTs of \( x[n_1 + N_1 n_2] \).
2. Multiply result by twiddle factors \( W_N^{n_1 k_2} \). Called twiddle multis.
3. Compute \( N_2 \) (for each \( k_2 \)) \( N_1 \)-point DFTs of the result. Done!

\( (N_1N_2) \text{-point} \rightarrow N_1 \text{(} N_2 \text{-point)} + N_2 \text{(} N_1 \text{-point)} + (N_1-1)(N_2-1) \text{twiddle.} \)

since: \( n_1 = 0 \) or \( k_2 = 0 \rightarrow W_N^{n_1 k_2} = 1. \) This is important below.

GENERAL DERIVATION [5/5]

**Visual:** \( (N_1 \times N_2) \) arrays: \( x_{n_1,n_2} = x[n_1 + N_1 n_2]; \) \( X_{k_1,k_2} = X_{k_2 + N_2 k_1}. \)

1. Take \( N_2 \)-point DFT of each row (fixed \( n_1 \)). Yields \( \hat{x}_{n_1,k_2}. \)
2. Multiply \( \hat{x}_{n_1,k_2} \) point-by-point by the twiddle factor \( W_N^{n_1 k_2}. \)
3. Take \( N_1 \)-point DFT of each column (fixed \( k_2 \)). Yields \( X_{k_1,k_2}. \)

**Example:** Later in lecture.

RADIX-2 COOLEY-TUKEY FFT

1. Decimation-in-time: \( N = 2^N/2 \).

**Set:** \( N_1 = 2; \) \( N_2 = N/2; \) \( n_1 = 0,1; k_1 = 0,1. \)

2. Decimation-in-freq: \( N = N_2/2. \)

**Set:** \( N_1 = N/2; \) \( N_2 = 2; \) \( n_2 = 0,1; k_2 = 0,1. \)

Then: \( N \text{-point} \rightarrow 2(N/2 \text{-point}) + N/2 (2 \text{-point}) \)

\(+ (N/2 - 1) \text{ twiddle multis. Total: } N/2 \log_2 N \text{ multis.} \)

**Note:** \( (N_1-1)(N_2-1)=(2-1)(N/2-1)=N-1; \)

**That is:** Half the twiddle multis are trivial!

DEC.-IN-TIME RADIX-2 COOLEY-TUKEY FFT

\[ X_{k_2} = \sum_{n_2=0}^{N/2-1} W_{N/2}^{n_2 k_2} x[2n_2] + W_{N}^{k_2} \sum_{n_2=0}^{N/2-1} W_{N/2}^{n_2 k_2} x[2n_2 + 1]. \]

\[ X_{k_2 + N/2} = \sum_{n_2=0}^{N/2-1} W_{N/2}^{n_2 k_2} x[2n_2] - W_{N}^{k_2} \sum_{n_2=0}^{N/2-1} W_{N/2}^{n_2 k_2} x[2n_2 + 1]. \]

\[ N/2 \text{-point DFT} \quad \text{N}/2 \text{-point DFT} \]

\[ N/2 \text{-point DFT} \quad \text{N}/2 \text{-point DFT} \]
DEC-IN-FREQ RADIX-2 COOLEY-TUKEY FFT

\[ X_{2k} = \sum_{n_1=0}^{N/2-1} W^{n_1 k_j}_{N/2}[x[n_1] + x[n_1 + N/2]]1. \]

\[ X_{2k+1} = \sum_{n_1=0}^{N/2-1} W^{n_1 k_j}_{N/2}[x[n_1] - x[n_1 + N/2]]W^{1n_1}_{N}. \]

COMPUTING \( F \) USING DFT [1/3]

Goal: Compute numerically

\( F: X(f) = \int x(t)e^{-j2\pi ft}dt. \)

\( F^{-1}: x(t) = \int X(f)e^{j2\pi ft}df. \)

Note: \( X(f) = X(\frac{f}{2N}) \) & \( df = \frac{2\pi}{2\pi}. \)

Assume: \( x(t) \) time-limited to \( 0 < t < T. \)

Assume: \( X(f) \) band-limited to \( |f| < B/2. \)

COMPUTING \( F \) USING DFT [2/3]

Sample t: \( t = n\Delta_t \) for \( 0 \leq n \leq N - 1. \)

Sample f: \( f = k\Delta_f \) for \( 0 \leq k \leq N - 1. \)

Nyquist: \( \Delta_t = 1/B \) and \( \Delta_f = 1/T. \)

Approx: \( X(f) \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi fn\Delta_t}\Delta_t. \)

Sample f: \( X(k\Delta_f) \approx \sum_{n=0}^{N-1} x(n\Delta_t)e^{-j2\pi nk\Delta_t}\Delta_f. \)

and: \( x(n\Delta_t) \approx \sum_{k=0}^{N-1} X(k\Delta_f)e^{j2\pi nk\Delta_t}\Delta_f. \)

COMPUTING \( F \) USING DFT: EXAMPLE

EX: \( F\{e^{-|t|}\} = \frac{2}{e^{\pi/4}} = \frac{2}{\pi}e^{\pi/4}. \)

Limits: \(-6 < t < 6 \rightarrow T = 12\)

and: \(-8 < f < 8 \rightarrow B = 16.\)

Values: \( N = BT = (16)(12) = 192; \)

and: \( \Delta_t = \frac{1}{B} = \frac{1}{16}; \Delta_f = \frac{1}{T} = \frac{1}{12}. \)

FFT: 192-point FFT of \( x(t) \) sampled \( t = n/16 \)

gives: \( 16X(f) \) sampled at \( f = k/12. \)

Result: Actual & computed coincide.