TOPICS FOR TODAY'S LECTURE

1. Discrete-Time Fourier Transform (DTFT)
   Relation to $X(z)$ and Fourier Transform $\mathcal{F}$
2. Frequency Response of LTI Systems
3. Notch (line reject) filters

Discrete-Time Fourier Transform (DTFT)

DEF: DTFT$\{x[n]\}$=$X(e^{j\omega})$ where

$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$.

1. $X(e^{j\omega})=X(z)|_{z=e^{j\omega}}$ (same $X(\cdot)$).
2. **DTFT is periodic with period=2π!**
   This is why I use $X(e^{j\omega})$, not $X(\omega)$.
   Usually compute/plot for $-\pi < \omega \leq \pi$.

DTFT vs. Fourier Transform

DTFT: $X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$.

$\mathcal{F}:$ $X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$.

DTFT: $X(e^{j\omega})=X(z)|_{z=e^{j\omega}}$.
$\mathcal{F}:$ $X(j\omega)=X(s)|_{s=j\omega}$.

Fact: $\mathcal{F}\{\sum x[n]\delta(t-n)\}$=DTFT$\{x[n]\}$.

DTFT vs. $\mathcal{Z}$ vs. $\mathcal{F}$ vs. $\mathcal{L}$

Continuous	$\mathcal{L}$	$\Leftrightarrow$	$\mathcal{F}$	Discrete xform
$z=e^s$	$\Updownarrow$	$s=j\omega$	$\Updownarrow$	from Continuous
Discrete	$\mathcal{Z}$	$\Leftrightarrow$	$DTFT$	$\mathcal{F}\{\sum x[n]\delta(t-n)\}$
Time		$z=e^{j\omega}$		$\mathcal{L}\{\sum x[n]\delta(t-n)\}$

Inverse DTFT vs. Fourier Series

DEF: $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$.
Note: Can use $\int_{-\pi}^{\pi}$ or $\int_{0}^{2\pi}$ or $\int_{p}^{p+2\pi}$

How: Period=2π $\rightarrow$expand $X(e^{j\omega})$ as a **Fourier series** with period 2π
where: $x[n]$=Fourier series coefficients; compute using $DTFT^{-1}$ formula.

DTFT: PROPERTIES

1. **Periodic with period=2π!**
2. If $x[n]$ real, have **conjugate symmetry:**
   a. $|X(e^{j\omega})|$ & real$[X(e^{j\omega})]$ even functions
   b. arg$[X(e^{j\omega})]$ & imag$[X(e^{j\omega})]$ odd funcs.
   c. $X(e^{-j\omega})=X(e^{j\omega})^*$
   d. DTFT[real & even]=real & even
3. DTFT[x[n–D]]=$X(e^{j\omega})e^{-j\omega D}$:
   Time delay$\rightarrow$linear phase shift.
4. DTFT[x[n]$e^{j\omega_o n}$]=$X(e^{j(\omega-\omega_o)})$:
   Modulation$\rightarrow$frequency shift

DTFT: EXAMPLES [1/2]

Easiest: Compute $X(z)$, then set $z = e^{j\omega}$.

EX #1: $x[n]=\{\ldots 0,0,3,1,\underline{4},2,5,0,0\ldots\}$

Soln: $X(e^{j\omega})=3e^{j2\omega}+1e^{j\omega}+4+2e^{-j\omega}+5e^{-j2\omega}$.

Answer: $X(e^{j\omega})=[4+3\cos(\omega)+8\cos(2\omega)]-j[\sin(\omega)+2\sin(2\omega)]$.

using: $Ae^{j\omega}+Be^{-j\omega}=(A+B)\cos(\omega)+j(A-B)\sin(\omega)$ (Euler)

Note: Conjugate symmetry in $X(e^{j\omega})$.

DTFT: APPLICATIONS

Q: So what is DTFT **for**?

A: Same things $\mathcal{F}$ is for:

1. Signal spectra (next time)
2. Frequency response (now)

FREQUENCY RESPONSE [2/2]

Gain: Amplitude increases by factor $|H(e^{j\omega_o})|$.

Phase: Shift by $\arg[H(e^{j\omega_o})] = \tan^{-1}\frac{\mathrm{Im}[H(e^{j\omega_o})]}{\mathrm{Re}[H(e^{j\omega_o})]}$.

DTFT: $H(e^{j\omega_o}) = \sum h[n]e^{-j\omega_o n} = DTFT[h[n]]$.

Zero: $H(z)$ has zero at $e^{\pm j\omega_o} \rightarrow y[n] = 0$ (in SS).

Pole: $H(z)$ has pole at $e^{\pm j\omega_o} \rightarrow y[n] \rightarrow \infty$.

DTFT: EXAMPLES [2/2]

EX #2: $x[n]=a^n u[n] \rightarrow X(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}$.

EX #3: $x[n]=a^n u[n]+b^n u[-n-1]$ (2-sided).

$\rightarrow X(e^{j\omega})=\frac{e^{j\omega}}{e^{j\omega}-a}-\frac{e^{j\omega}}{e^{j\omega}-b} = \frac{b-a}{a+b-e^{j\omega}-abe^{-j\omega}}$.

Need: $|a| < 1 < |b|$ or DTFT does not exist!

FREQUENCY RESPONSE [1/2]

Recall: $z_o^n \rightarrow \boxed{\mathbf{H(z)}} \rightarrow H(z_o)z_o^n$. Eigenfunction of LTI.

Now: $z_o^n = e^{+j\omega_o n} \rightarrow \boxed{\mathbf{H(z)}} \rightarrow H(e^{+j\omega_o})e^{+j\omega_o n}$.

and: $z_o^n = e^{-j\omega_o n} \rightarrow \boxed{\mathbf{H(z)}} \rightarrow H(e^{-j\omega_o})e^{-j\omega_o n}$.

using: $H(e^{\pm j\omega_o}) = |H(e^{+j\omega_o})|\exp(\pm j\arg(H(e^{+j\omega_o})))$

$\cos(\omega_o n) \rightarrow \boxed{\mathbf{H(z)}} \rightarrow |H(e^{j\omega_o})|\cos(\omega_o n + \arg[H(e^{j\omega_o})])$.

FREQUENCY RESPONSE: EX #1

System: $y[n]-\frac{1}{2}y[n-1]=x[n] \rightarrow h[n]=(\frac{1}{2})^n u[n]$.

Input: $x[n]=\{\ldots -1,0,\underline{1},0,-1\ldots\}=\cos(\frac{\pi n}{2})$.

$\omega = \frac{\pi}{2}$: $H(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}}=\frac{1}{1+\frac{j}{2}}=0.89e^{-j26.6^o}$.

Answer: $y[n]=0.89\cos(\frac{\pi n}{2}-26.6^o)$

$=\{\ldots 0.8, 0.4, -0.8, -0.4, 0.8, 0.4\ldots\}$.

FREQUENCY RESPONSE: EX #2

System: $y[n]-\frac{1}{2}y[n-1]=x[n] \rightarrow h[n]=(\frac{1}{2})^n u[n]$.

Input: $x[n]=\{\ldots-1,\underline{1},-1,1,-1\ldots\}=\cos(\pi n)$.

$\omega=\pi$: $H(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}}=\frac{1}{1+\frac{1}{2}}=\frac{2}{3}$.

Answer: $y[n]=\frac{2}{3}\cos(\pi n)=\frac{2}{3}(-1)^n$.

FREQUENCY RESPONSE: EX #3

System: $y[n]=\frac{1}{2}(x[n]+x[n-1]) \rightarrow h[n]=\{\underline{\tfrac{1}{2}},+\frac{1}{2}\}$.

$H(e^{j\omega})$: $\frac{1}{2}(1+e^{-j\omega})=\cos(\frac{\omega}{2})e^{-j\omega/2}$. Low-pass.

System: $y[n]=\frac{1}{2}(x[n]-x[n-1]) \rightarrow h[n]=\{\underline{\tfrac{1}{2}},-\frac{1}{2}\}$.

$H(e^{j\omega})$: $\frac{1}{2}(1-e^{-j\omega})=\sin(\frac{\omega}{2})e^{j(\pi-\omega)/2}$. High-pass.

FREQUENCY RESPONSE: EX #4

System: $y[n]=x[n+1]-2\cos(\omega_o)x[n]+x[n-1]$.

h[n]: $\{1,\underline{-2\cos(\omega_o)},1\}$.

H(z): $z-2\cos(\omega_o)+z^{-1}=(z-e^{j\omega_o})(z-e^{-j\omega_o})\frac{1}{z}$.

$H(e^{j\omega})$: $e^{j\omega}-2\cos(\omega_o)+e^{-j\omega}=2\cos(\omega)-2\cos(\omega_o)$.

Does: This filter rejects $\omega=\omega_o$.

Name: *Notch* or line-reject filter.

Delay: Make causal: output delayed by one.

NOTCH FILTER: EXAMPLE #1

Given: Lab sensors with wire leads

Given: DSP system with rate $360\frac{\text{SAMPLE}}{\text{SECOND}}$.

But: 60 Hertz interference from walls

Goal: Design digital filter to eliminate.

Soln: Notch filter with $\omega_o=2\pi\frac{60}{360}$.

Need: $h[1]=-2\cos(2\pi\frac{60}{360})=-1$

Answer: $y[n]=x[n]-x[n-1]+x[n-2]$.

NOTCH FILTER: EXAMPLE #2 [1/2]

Goal: Filter to reject both 60 Hz and 120 Hz.

Soln: Use 2 notch filters in series or cascade.

Need: $-2\cos(2\pi\frac{60}{360})=-1$ and $-2\cos(2\pi\frac{120}{360})=1$.

Soln: $h[n]=\{\underline{1},-1,1\}*\{\underline{1},1,1\}=\{\underline{1},0,1,0,1\}$.

Answer: $y[n]=x[n]+x[n-2]+x[n-4]$.

NOTCH FILTER: EXAMPLE #2 [2/2]

Note the above $H(z)=\frac{z^4+z^2+1}{z^4}$ has
Zeros $e^{\pm j2\pi(60/360)}$ and $e^{\pm j2\pi(120/360)}$.
Recall zeros at $e^{\pm j\omega_o}$ rejects $A\cos(\omega_o n+\theta)$.

Alternative: Design filter to have these zeros:
$H(z)=(z-e^{j2\pi\frac{60}{360}})(z-e^{-j2\pi\frac{60}{360}})(z-e^{j2\pi\frac{120}{360}})(z-e^{-j2\pi\frac{120}{360}})$
$H(z)=z^4+z^2+1$. Divide by $z^4 \Leftrightarrow$ delay by 4.