TOPICS FOR TODAY'S LECTURE

1. Discrete-Time Fourier Series (DTFS)
   DTFS computation examples
2. Frequency Response and DTFS
   Frequency response examples

CONTINUOUS-TIME FOURIER SERIES

Given: $x(t) = x(t+T)$ periodic; period=T.

Then: $x(t)$ can be expanded as

Series: $x(t)=x_0+x_1 e^{j\frac{2\pi}{T}t}+x_2 e^{j\frac{4\pi}{T}t}+\ldots$

$+x_{-1}e^{-j\frac{2\pi}{T}t}+x_{-2}e^{-j\frac{4\pi}{T}t}+\ldots$

where: $x_k=\frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi}{T}kt}dt$ for integers k.

Note: **Conjugate symmetry:** $x(t)$ real$\Leftrightarrow$ $x_{-k}=x_k^*$.

DTFS PROPERTIES

1. $x_0=\frac{1}{N}\sum_{n=0}^{N-1} x[n]$=DC value=mean value of $x[n]$.
2. N even: Highest frequency component:
   $x_{N/2}=\frac{1}{N}$(x[0]–x[1]+x[2]–x[3]+…–x[N–1]).
3. $k<0=2^{nd}$ half of $\{x_k\}$: $x_{-k}=x_{N-k}$.
4. Matlab: `fft(X,N)/N` computes $\{x_0 \ldots x_{N-1}\}$.
5. Matlab's `fftshift` shifts DC to center.
6. Parseval: $\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2=\sum_{k=0}^{N-1}|x_k|^2$.

DISCRETE-TIME FOURIER SERIES (DTFS)

Given: $x[n]=x[n+N]$ periodic; period=N.

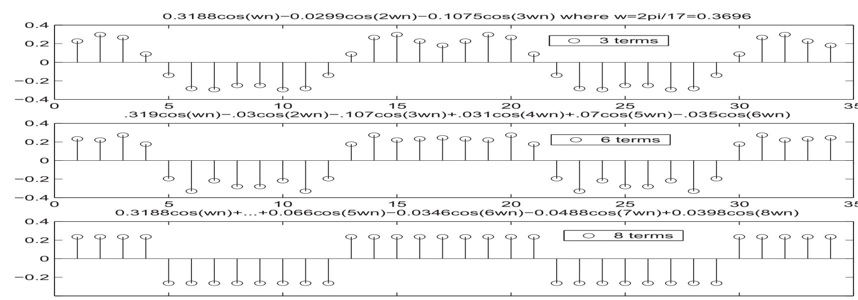
Then: $x[n]$ can be expanded as

DTFS: $x[n]=x_0+x_1 e^{j\frac{2\pi}{N}n}+x_2 e^{j\frac{4\pi}{N}n}+\ldots+x_{N-1}e^{j\frac{(N-1)2\pi}{N}n}$

where: $x_k=\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$ for $k=0\ldots N-1$.

Note: **Conjugate symmetry:** $x[n]$ real$\Leftrightarrow$ $x_{N-k}=x_k^*$.

Square: $x(n)=\begin{cases}1 & \text{if } 0\le n\le L-1\\ 0 & \text{if } L\le n\le N-1\end{cases} \rightarrow x_k=\begin{cases}\frac{L}{N} & \text{if N divides k; else}\\ \frac{1}{N}\frac{\sin(\pi kL/N)}{\sin(\pi k/N)}e^{-j\pi k(L-1)}\end{cases}$



COMPUTING DTFS: EXAMPLE #1 [1/3]

Given: $x[n]=\{\ldots 12,6,4,6,\underline{12},6,4,6,12,6,4,6\ldots\}$.

Goal: Compute its DTFS. **By inspection:** $N=4$.

Note: $e^{-j\frac{2\pi}{4}1}$=–j; $e^{-j\frac{2\pi}{4}2}$=–1; $e^{-j\frac{2\pi}{4}3}$=+j.

$x_0=$ $\frac{1}{4}$(x[0]+(+1)x[1]+(+1)x[2]+(1)x[3])=$\frac{1}{4}$(12+6+4+6)=7.

$x_1=$ $\frac{1}{4}$(x[0]+(–j)x[1]+(–1)x[2]+(+j)x[3])=$\frac{1}{4}$(12–6j–4+6j)=2.

$x_2=$ $\frac{1}{4}$(x[0]+(–1)x[1]+(+1)x[2]+(–1)x[3])=$\frac{1}{4}$(12–6+4–6)=1.

$x_3=$ $\frac{1}{4}$(x[0]+(+j)x[1]+(–1)x[2]+(–j)x[3])=$\frac{1}{4}$(12+6j–4–6j)=2.

Note: $x[n]$ real & even function$\Leftrightarrow$ x_k real & even.

COMPUTING DTFS: EXAMPLE #1 [2/3]

Then: $x[n]=7+2e^{j\frac{\pi}{2}n}+1e^{j\pi n}+2e^{j\frac{3\pi}{2}n}$

Or: $x[n]=7+4\cos(\frac{\pi}{2}n)+1\cos(\pi n)$.

since: $e^{j\frac{3\pi}{2}n}=e^{j(2\pi-\frac{\pi}{2})n}=e^{-j\frac{\pi}{2}n}$

and: $e^{j\pi n}=\cos(\pi n)=(-1)^n$.

COMPUTING DTFS: EXAMPLE #1 [3/3]

Parseval: Average power same in both domains:

Time: $\frac{1}{4}(12^2+6^2+4^2+6^2) \quad =58=\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$.

Freq.: $(|7|^2+|2|^2+|1|^2+|2|^2)=58=\sum_{k=0}^{N-1}|x_k|^2$ (NO $\frac{1}{N}$).

Note: Avg. power of $e^{j2\pi kn/N}$ is $\frac{1}{N}\sum_{n=0}^{N-1}|e^{j2\pi kn/N}|^2=1$.

COMPUTING DTFS: EXAMPLE #2 [1/2]

Given: $x[n]=\{\ldots 24,8,12,16,\underline{24},8,12,16,24,8,12,16\ldots\}$.

Goal: Compute its DTFS. **By inspection:** $N=4$.

$x_0=$ $\frac{1}{4}$(x[0]+(+1)x[1]+(+1)x[2]+(1)x[3])=$\frac{1}{4}$(24+8+12+16)=15.

$x_1=$ $\frac{1}{4}$(x[0]+(–j)x[1]+(–1)x[2]+(+j)x[3])=$\frac{1}{4}$(24–8j–12+16j)=3+j2.

$x_2=$ $\frac{1}{4}$(x[0]+(–1)x[1]+(+1)x[2]+(–1)x[3])=$\frac{1}{4}$(24–8+12–16)=3.

$x_3=$ $\frac{1}{4}$(x[0]+(+j)x[1]+(–1)x[2]+(–j)x[3])=$\frac{1}{4}$(24+8j–12–16j)=3–j2.

Note: $x[n]$ real$\rightarrow x_k=x^*_{4-k}$.

COMPUTING DTFS: EXAMPLE #2 [2/2]

Then: $x[n]=(15)e^{j0n}+(3+2j)e^{j(\pi/2)n}+(03)e^{j\pi n}+(3-2j)e^{j(3\pi/2)n}$

Or: $x[n]=15+7.2\cos(\frac{\pi}{2}n+33.7^o)+3\cos(\pi n)$ since 3+j2=$3.6e^{j33.7^o}$.

Time: $\frac{1}{4}(24^2+8^2+12^2+16^2) \quad =260=\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$.

Freq.: $(|15|^2+|3+j2|^2+|3|^2+|3-j2|^2)=260=\sum_{k=0}^{N-1}|x_k|^2$.

COMPUTING DTFS: EXAMPLE #3 [1/1]

Goal: Compute DTFS$\{\cos(2\pi\frac{M}{N}n+\theta)\}$.

Soln: $\cos(2\pi\frac{M}{N}n+\theta)=\frac{1}{2}e^{j2\pi\frac{M}{N}n}e^{j\theta}+\frac{1}{2}e^{-j2\pi\frac{M}{N}n}e^{-j\theta}$

Answer: $\frac{1}{2}e^{j\theta}\delta[k-M]+\frac{1}{2}e^{-j\theta}\delta[k-(N-M)]$.

Only: *Periodic* discrete-time sinusoids: $\omega_o=2\pi\frac{M}{N}$.

FREQ. RESPONSE: EX #1 [1/3]

System: y[n]–3y[n–1]=3x[n]+3x[n–1].

Input: x[n]=$\{\ldots 12,6,4,6,\underline{12},6,4,6\ldots\}$.

Goal: Compute the output $y[n]$.

Plan: Compute DTFS expansion of x[n].

Then: Find effect of system on each term .

FREQ. RESPONSE: EX #1 [2/3]

System: y[n]–3y[n–1]=3x[n]–3x[n–1].

$H(z) = \frac{Y(z)}{X(z)} = \frac{3-3z^{-1}}{1-3z^{-1}}\frac{z}{z} = 3\frac{z-1}{z-3}$.

$H(e^{j\omega}) = 3\frac{e^{j\omega}-1}{e^{j\omega}-3}$. Plug in frequencies:

$H(e^{j0}) = 3\frac{+1-1}{+1-3} = 0$; $H(e^{j\pi/2}) = 3\frac{+j-1}{+j-3} = 1.341e^{-j0.46}$.

$H(e^{j\pi}) = 3\frac{-1-1}{-1-3} = \frac{3}{2}$; $H(e^{j3\pi/2}) = 3\frac{-j-1}{-j-3} = 1.341e^{j0.46}$.

FREQ. RESPONSE: EX #1 [3/3]

Input: $x[n]=\{\ldots 12, 6, 4, 6, \underline{12}, 6, 4, 6 \ldots\}$.

DTFS: $x[n]=7+2e^{j\frac{\pi}{2}n}+1e^{j\pi n}+2e^{j\frac{3\pi}{2}n}$

Output: $y[n]= [0]7+[1.341e^{-j0.46}][2e^{jn\pi/2}]$
$+[\frac{3}{2}][1e^{jn\pi}]+[1.341e^{j0.46}][2e^{jn3\pi/2}]$

Becomes: $y[n]=5.366\cos(\frac{\pi}{2}n-0.46)+1.5\cos(\pi n)$.

OR: $x[n]=7+4\cos(\frac{\pi}{2}n)+1\cos(\pi n)$.

Output: $[0]7+[1.341]4\cos(\frac{\pi}{2}n-0.46)+[\frac{3}{2}]1\cos(\pi n)$.

FREQUENCY RESPONSE: EX #2 [1/4]

System: $y[n]=\frac{1}{2}(x[n]+x[n-1])$ (average most recent).

Input: $x[n]=\{\ldots 4, 0, 1, 0, 1, 0, \underline{4}, 0, 1, 0, 1, 0 \ldots\}$.

Goal: Compute the output y[n].

$H(e^{j\omega}) = \frac{1}{2}+\frac{1}{2}e^{-j\omega} = (\cos\frac{\omega}{2})e^{-j\omega/2}$ from before.

FREQUENCY RESPONSE: EX #2 [2/4]

Matlab: `fft([4 0 1 0 1 0],6)/6`

Get: `[1 .5 .5 1 .5 .5].`

DTFS: $x[n]=1+.5e^{j2\pi n/6}+.5e^{j4\pi n/6}$
$+1e^{j6\pi n/6}+.5e^{j8\pi n/6}+.5e^{j10\pi n/6}$.

Simplify: $1+\cos(\frac{\pi}{3}n)+\cos(\frac{2\pi}{3}n)+\cos(\pi n)$.

Parseval: Useful as a check on DTFS:

Time: $\frac{1}{6}(4^2+0^2+1^2+0^2+1^2+0^2)=3$.

Freq.: $1^2+(\frac{1}{2})^2+(\frac{1}{2})^2+1^2+(\frac{1}{2})^2+(\frac{1}{2})^2=3$.

FREQUENCY RESPONSE: EX #2 [3/4]

$x[n]$:	1	$\cos(\frac{\pi}{3}n)$	$\cos(\frac{2\pi}{3}n)$	$\cos(\pi n)$
ω :	0	$\pi/3$	$2\pi/3$	π
$H(e^{j\omega})$:	1	$0.866\angle-\frac{\pi}{6}$	$0.5\angle-\frac{\pi}{3}$	0
Gain :	1	0.866	0.5	0
Phase :	0	$-\pi/6$	$-\pi/3$	NA

FREQUENCY RESPONSE: EX #2 [4/4]

Answer: $1+0.866\cos(\frac{\pi}{3}n-\frac{\pi}{6})+0.5\cos(\frac{2\pi}{3}n-\frac{\pi}{3})+0$

Simplify: $\{\ldots 2, 2, .5, .5, .5, .5, \underline{2}, 2, .5, .5, .5, .5 \ldots\}$.

Note: Higher frequencies of x[n] reduced.

Thus: System smoothes the input signal.