DISCRETIZING LCCDEs TO DIFFERENCE EQNS.

1. Backward difference approximation
2. Frequency response of backward difference
3. z-transform for backward difference approximation to higher derivatives
4. LCCDE to ARMA difference equation
5. Simple numerical example

DISCRETIZING LCCDEs TO DIFFERENCE EQNS.

Given: An LCCDE:
Linear Constant-Coefficient Differential Equation
Goal: Discretize LCCDE to a difference equation
So: Can compute solution recursively and quickly
Why? Many applications in engineering branches
Note: Coefficients are different from LCCDE ones

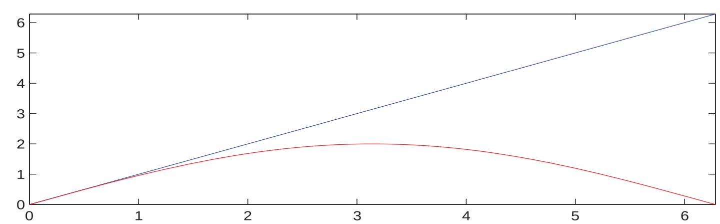
BACKWARD DIFFERENCE APPROXIMATION [1/2]

Taylor $x(t{-}T)=x(t){-}T\frac{dx}{dt}(t)+\frac{T^2}{2!}\frac{d^2x}{dt^2}(t)+\ldots\approx x(t){-}T\frac{dx}{dt}(t)$.
Series Becomes $\frac{dx}{dt}(t)\approx\frac{x(t)-x(t-T)}{T}$. But when is this valid?

Fourier $\mathcal{F}\{x(t)\}=X(j\Omega)$ where $\Omega=2\pi f$ where f is in Hertz.
Frequency Read off $H(j\Omega)_{\text{back}}=\frac{1-e^{-j\Omega T}}{T}\approx j\Omega=H(j\Omega)_{\text{ideal}}$
Response if $\Omega T<<1$. Easier than $|T\frac{dx}{dt}(t)|>>|\frac{T^n}{n!}\frac{d^nx}{dt^n}(t)|, n=2,3\ldots$

Next $|H(j\Omega)_{\text{ideal}}|$ in blue vs. ΩT. `W=linspace(0,2*pi,1000);`
Slide $|H(j\Omega)_{\text{back}}|$ in red vs. ΩT. `H1=W;H2=abs(1-exp(j*W));`

BACKWARD DIFFERENCE APPROXIMATION [2/2]



SAMPLING [1/2]

Sample: $x[n]=x(nT)$ and $y[n]=y(nT)$ will work well
If: $x(t)$ and $y(t)$ are bandlimited to $\Omega<<\frac{1}{T}$.

Backward $y[n]=(x[n]{-}x[n{-}1])/T$. $h[n]=\{\underline{\tfrac{1}{T}},-\tfrac{1}{T}\}$.
Difference $H(z)=(1{-}z^{-1})/T$. $H(e^{j\omega})=(1{-}e^{-j\omega})/T$. $\omega=\Omega T$.

$\frac{d^2x}{dt^2}(t)\approx\left(\frac{dx}{dt}(t)-\frac{dx}{dt}(t-T)\right)/T$. Substitute in itself.
$h[n]=\frac{1}{T}(\frac{1}{T}\{\underline{1},-1\}{-}\frac{1}{T}\{\underline{0},1,-1\})=\frac{1}{T^2}(\{\underline{1},-2,1\})$.
$\frac{d^3x}{dt^3}(t)\approx\left(\frac{d^2x}{dt^2}(t)-\frac{d^2x}{dt^2}(t-T)\right)/T$. $h[n]$: gets complicated.

SAMPLING [2/2]

Easier: $\frac{d^Nx}{dt^N}$ approximated by series connection.
Transfer: $H(z)^N=\left(\frac{1}{T}(1-z^{-1})\right)^N$. Easy to expand.

Given: LCCDE $\sum_{i=0}^N a_{N-i}\frac{d^iy}{dt^i}=\sum_{i=0}^M b_{M-i}\frac{d^ix}{dt^i}$.
Substitute: $\frac{d^ix}{dt^i}\rightarrow\left(\frac{1}{T}(1-z^{-1})\right)^i$ everywhere.

Get: $\sum_{i=0}^N a_{N-i}\left(\frac{1-z^{-1}}{T}\right)^i Y(z)=\sum_{i=0}^M b_{M-i}\left(\frac{1-z^{-1}}{T}\right)^i X(z)$.
Collect: Terms of coefficients of powers of z^{-i}.
Read off: ARMA difference equation coefficients.

EXAMPLE [1/4]

Given: $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y(t)=2e^{-3t}u(t)$. Initial conditions=0.
Goal: Discretize this LCCDE into ARMA difference equation
Compute: Solution to latter numerically by iteration.
Use: T=0.001 to make this easy to follow.

Right Side: $x(t)=2e^{-3t}u(t)$. Discretize this to $x[n]=x(0.001n)$:
$x[n]=2e^{-0.003n}u[n] \rightarrow X(z)=\frac{2z}{z-e^{-0.003}}$.

Left $\left(\frac{1-z^{-1}}{T}\right)=1000(1-z^{-1})$. $\left(\frac{1-z^{-1}}{T}\right)^2=10^6(1-2z^{-1}+z^{-2})$.

EXAMPLE [2/4]

Sub: $[10^6(1-2z^{-1}+z^{-2})+3(10^3)(1-z^{-1})+2]Y(z)=\frac{2z}{z-e^{-0.003}}$.

Collect Terms: $[10^6(1-2z^{-1}+z^{-2})+3(10^3)(1-z^{-1})+2]Y(z)=$
$[z^0(1003002)-z^{-1}(2003000)+z^{-2}(10^6)]Y(z)=\frac{2z}{z-e^{-0.003}}$.

Read off: $(1003002)y[n]-(2003000)y[n-1]+(10^6)y[n-2]=2e^{-0.003n}u[n]$.

Recursive: $y[n]=\frac{2e^{-0.003n}}{1003002}+\frac{2003000}{1003002}y[n-1]-\frac{1000000}{1003002}y[n-2]$. Set n=2,3,4...

EXAMPLE [3/4]

Actual: $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y(t)=2e^{-3t}u(t)$. Initial conditions=0.
Laplace: $Y(s)[s^2+3s+2]=\mathcal{L}\{2e^{-3t}u(t)\}=2/(s+3)$.

Solve: $Y(s)=\frac{2}{(s^2+3s+2)(s+3)}=\frac{1}{s+1}-\frac{2}{s+2}+\frac{1}{s+3}$.
Get: $y(t)=[e^{-t}-2e^{-2t}+e^{-3t}]u(t)$. Scale everything by 10^6:

```
T=linspace(0,6,6000);X=exp(-T)-2*exp(-2*T)+exp(-3*T);
B=[0.000002];A=[1.003002 -2.003 1];
Y=filter(B,A,exp(-3*T));plot(T,X,T,Y,'r--')
```

EXAMPLE [4/4]