TOPICS FOR TODAY'S LECTURE
1. Auto-correlation of a signal
   Use: Estimation of unknown period of periodic signal
2. Cross-correlation of 2 signals
   Use: Estimation of unknown time-delay of delayed signal
3. Signal similarity of 2 signals
   Use: Which one of several possible signals is present?
Assume: $x[n]$ is real-valued throughout.

AUTO-CORRELATION [1/3]
DEF: Auto-correlation $r_x[n]$ of $x[n]$: $r_x[n]=x[n] \ast x[-n]=\sum_{i=-\infty}^{\infty} x[i]\cdot x[i+n]=\sum_{i=-\infty}^{\infty} x[i]\cdot x[i-n]$.
Note: $n$ is called the lag; NOT time (but same units).
DTFT: $R_x(e^{j\omega})=X(e^{j\omega})X(e^{-j\omega})=X(e^{j\omega})\cdot X(e^{-j\omega})^* = |X(e^{j\omega})|^2$.
Given: $x[n]$ stored in row vector X. Compute using Matlab:
Then: $L=\text{length}(X); N=\text{nextpow2}(2*L-1)$;
     and: $R=\text{real}(	ext{ifft}(|\text{fft}(X,N)|^{-2}))$;

AUTO-CORRELATION [2/3]
PROPERTIES OF $r_x[n]=x[n] \ast x[-n]$:
1. $r_x[n]=r_x[-n]$ ($r_x[n]$ is an even function).
2. $r_x[0] \geq r_x[n]$ for any lag $n$; usually $r_x[0] >> r_x[n]$.
3. If $|x[n]|$ has length $N$, then $r_x[n]$ has length $=2N-1$.
4. $r_x[n] \neq 0$ for $-(N-1) \leq n \leq (N-1)$.
5. Delay or advance $x[n] \rightarrow$ no change in $r_x[n]$.

AUTO-CORRELATION [3/3]
EXAMPLE: COMPUTING $r_x[n]$:
Given: $x[n]=\{3, 1, 4\}$. Location of $n=0$ irrelevant!
$r_x[n]=\{3 \cdot 4, 3 \cdot 1+1 \cdot 4, 3^2+4^2, 3 \cdot 1+1 \cdot 4, 3 \cdot 4\}$
$r_x[0]=\{3, 1, 4 \cdot 4, 1, 3\}=\{12, 7, 26, 7, 12\}$.
Note: $r_x[0]=\sum x[n]^2=\text{energy}=3^2+1^2+4^2=26$.
Note: $r_x[n]$ is even function; $r_x[0] > r_x[n]$ for $n \neq 0$.

WHITE GAUSSIAN NOISE (WGN)
DEF: $w[n]=0$-mean white Gaussian noise with strength $\sigma^2$.
Means: $w[i]$ and $w[j]$ are independent for $i \neq j$.
And: $\Pr[-\sigma \leq w[n] \leq \sigma] \approx \frac{2}{\sqrt{\pi}}$ since Gaussian distribution.
Get: $r_w[n] \approx \sigma^2 \delta[n]$ and $r_{wx}[n]=w[n] \ast x[-n] \approx 0$.
Take: EECS 301 for more details on WGN.

ESTIMATION OF UNKNOWN PERIOD [1/6]
Let: $x[n]$ be periodic with unknown period $N$.
Given: $y[n]=x[n]+w[n]$ (noisy observations of $x[n]$)
where: $w[n]=0$-mean white Gaussian noise with strength $\sigma^2$.
Goal: Estimate $N$ from noisy observations $y[n]$.
Why? $x[n]$ is sound of a musical instrument.
Goal: What musical note is being played by it?
### ESTIMATION OF UNKNOWN PERIOD [2/6]


**b**ecomes: \( r_y[n] = r_x[n] + r_w[n] = r_x[n] + \sigma^2 \delta[n] \)

**s**ince: \( w[n] * x[-n] = x[n] * w[-n] = 0 \) by assumption (usually true).

**N**ow: \( r_x[N] = \sum_i x[i] x[i+N] = \sum_i x[i]^2 = r_x[0] \)

**s**ince: \( x[n] \) is periodic with period=\( N \).

**A**lso: \( r_x[0] = r_x[N] = r_x[2N] = \ldots \gg r_x[n] \) for other \( n \).

### ESTIMATION OF UNKNOWN PERIOD [3/6]

**S**o: **T**O **E**STIMATE \( N \) **F**ROM \( y[n] \):

1. Compute \( r_y[n] \) from given \( y[n] \) using FFT.
2a. \( r_y[0] = \sigma^2 + r_x[0] \) is larger than:
2b. \( r_y[N] = r_y[2N] = r_y[3N] = \ldots = r_y[0] \)
2c. \( r_y[n] < r_y[N] = r_y[2N] = r_y[3N] = \ldots \) for other \( n \).
3a. Look for 1st peak in \( r_y[n] \) with height \( \approx r_y[0] - \sigma^2 \).
3b. Location of this 1st peak is period \( N \) of \( x[n] \).

### ESTIMATION OF UNKNOWN PERIOD [4/6]

**E**X: Noise added to signal of trumpet playing unknown note.

**N**ext: Plots: noiseless trumpet; noisy trumpet; autocorrelation.

**G**et: \( y[n] \) has period=90 (not 45–look at it carefully).

**S**o: Frequency=\( \frac{44100 \text{ SAMPLES}}{\text{SECOND}} / 90 = 490 \text{ Hz} \)=note “B.”

### ESTIMATION OF UNKNOWN PERIOD [5/6]

![Graph showing the cross-correlation between two signals]

### CROSS-CORRELATION [1/2]

**D**EF: Cross-correlation \( r_{xy}[n] \) of \( x[n] \) and \( y[n] \):

\[ r_{xy}[n] = x[n] * y[-n] = \sum_{i=-\infty}^{\infty} x[i] y[i-n]. \] Also, 

\[ r_{yx}[n] = y[n] * x[-n] = \sum_{i=-\infty}^{\infty} x[i] y[i+n] = r_{xy}[-n]. \]

**D**TFT: 

\[ R_{xy}(e^{j\omega}) = X(e^{j\omega})Y(e^{-j\omega}) = X(e^{j\omega})Y(e^{j\omega})^*. \]

**G**iven: \( x[n], y[n] \) stored in row vectors \( X, Y \). Compute using Matlab:

**T**hen: \( L=\text{length}(X,Y); N=\text{nextpow2}(L-1) ; \)

\( \text{and: } R=\text{real}(\text{ifft}(\text{fft}(X,N).*\text{conj}(\text{fft}(Y,N)))); \)
TIME DELAY ESTIMATION [1/3]
Given: Observe $y[n] = x[n-D] + w[n]$
where: $x[n]$ is delayed by an unknown delay $D$, and: $w[n]$ is 0-mean white Gaussian noise with strength $\sigma^2$.
Goal: Estimate $D$ from noisy observations $y[n]$.

Why? $x[n]$ is a known radar or sonar pulse.
Goal: How far away is airplane or whale or sub?
Then: Distance $= Dc/2$ where $c$ is wave speed.

TIME DELAY ESTIMATION [2/3]

Given: $x[n] = \{3, 1, 4\}$ and $y[n] = \{2, 7, 1\}$.
$r_{xy}[n] = \{3 \cdot 1, 3 \cdot 7 + 1 \cdot 13 \cdot 2 + 1 \cdot 7 + 4 \cdot 1 \cdot 7 + 1 \cdot 2, 4 \cdot 2\}$
$r_{yx}[n] = \{3, 1, 4 \cdot \{1, 7, 2\} = \{3, 22, 17, 30, 8\}$.

Note: $r_{xy}[0] = \sum_i x[i][y[i]] = X \cdot Y$ (dot product).

Note: $Autocorrelation = \text{cross-correlation with itself}$.

TIME DELAY ESTIMATION [3/3]

Now: $r_x[n]$ is even, and usually $r_x[0] \gg r_x[n]$ for $n \neq 0$.
So: $r_{yx}[n]$ has a peak at $n = D =$ unknown delay.

1. Compute $r_{yx}[n] = y[n] \cdot x[-n] = \sum_{i=-\infty}^{\infty} x[i]y[i+n]$
2. Look for peak. Location = unknown delay $D$.

SIGNAL SELECTION

Given: $y[n] = \begin{cases} x_1[n - D_1] + \text{noise} & \text{OR} \\ x_2[n - D_2] + \text{noise} & \text{OR} \ldots \end{cases}$

Goal: Determine which one of the $x_i[n]$ is present.

Idea: Cross-correlate $y[n]$ with each $x_i[n]$.

Then: One of the $r_{yx_i}[n]$ will be large at $n = D_i$.
So: That $x_i[n]$ is the signal that is present.

Note: Works best when $x_i[n]$ are orthogonal (next slide).

SIGNAL SIMILARITY

DEF: $\rho = \frac{\sum_i x[i]y[i]}{\sqrt{\sum_i x[i]^2} \sqrt{\sum_i y[i]^2}} = \text{CORRELATION COEFFICIENT}$.

Then: $\rho = 1 \iff y[n] = ax[n], a \geq 0 \text{ and } \rho = -1 \iff y[n] = -ax[n]$.

EX: $x[n] = A \cos(\frac{2\pi}{N} n + \theta)$ and $y[n] = B \cos(\frac{2\pi}{N} n + \phi)$.

Then: $\rho = \cos(\theta - \phi)$ after much algebra.

EX: 180° phase difference $\iff \rho = -1 \iff$ multiply by $-B/A$.
And: 90° phase difference $\iff \rho = 0 \iff$ orthogonal (sin and cos).