TOPICS FOR TODAY'S LECTURE

1. Auto-correlation of a signal

Use: Estimation of unknown period of periodic signal

2. Cross-correlation of 2 signals

Use: Estimation of unknown time-delay of delayed signal

3. Signal similarity of 2 signals

Use: Which one of several possible signals is present?

Assume: $x[n]$ is real-valued throughout.

AUTO-CORRELATION [1/3]

DEF: Auto-correlation $r_x[n]$ of $x[n]$:

$r_x[n]=$ $x[n]*x[-n]=\sum_{i=-\infty}^{\infty} x[i]x[i+n]=\sum_{i=-\infty}^{\infty} x[i]x[i-n]$.

Note: n is called the *lag*; NOT *time* (but same units).

DTFT: $R_x(e^{j\omega})=X(e^{j\omega})X(e^{-j\omega})=X(e^{j\omega})X(e^{j\omega})^*=|X(e^{j\omega})|^2$.

Given: $x[n]$ stored in row vector `X`. Compute using Matlab:

Then: `L=length(X);N=nextpow2(2*L-1);`

and: `R=real(ifft(abs(fft(X,N)).^2));`

AUTO-CORRELATION [2/3]

PROPERTIES OF $r_x[n]=x[n]*x[-n]$:

1. $r_x[n]=r_x[-n]$ ($r_x[n]$ is an even function).
2. $r_x[0] \geq r_x[n]$ for any *lag* n; usually $r_x[0] >> r_x[n]$.
3. If $x[n]$ has length=N, then $r_x[n]$ has length=$2N$–1.
4. $r_x[n] \neq 0$ for $-(N\text{–}1) \leq n \leq (N\text{–}1)$.
5. Delay or advance $x[n]$ →no change in $r_x[n]$.

AUTO-CORRELATION [3/3]

EXAMPLE: COMPUTING $r_x[n]$:

Given: $x[n]=\{3,1,4\}$. Location of $n=0$ irrelevant!

$r_x[n]=$ $\{3\cdot 4, 3\cdot 1+1\cdot 4, 3^2+1^2+4^2, 3\cdot 1+1\cdot 4, 3\cdot 4\}$

$r_x[n]=$ $\{3,1,4\}*\{4,1,3\}=\{12,7,\underline{26},7,12\}$.

Note: $r_x[0]=\sum x[n]^2$=energy=$3^2+1^2+4^2$=26.

Note: $r_x[n]$ is even function; $r_x[0] > r_x[n]$ for $n \neq 0$.

WHITE GAUSSIAN NOISE (WGN)

DEF: $w[n]$=0-mean white Gaussian noise with strength σ^2:

Means: $w[i]$ and $w[j]$ are independent for $i \neq j$,

And: $\Pr[-\sigma \leq w[n] \leq \sigma] \approx \frac{2}{3}$ since Gaussian distribution.

Get: $r_w[n] \approx \sigma^2\delta[n]$ and $r_{wx}[n]=w[n]*x[-n] \approx 0$.

Take: EECS 301 for more details on WGN.

ESTIMATION OF UNKNOWN PERIOD [1/6]

Let: $x[n]$ be periodic with *unknown* period N.

Given: $y[n]=x[n]+w[n]$ (noisy observations of $x[n]$)

where: $w[n]$=0-mean white Gaussian noise with strength σ^2.

Goal: Estimate N from noisy observations $y[n]$.

Why? $x[n]$ is sound of a musical instrument.

Goal: What musical note is being played by it?

ESTIMATION OF UNKNOWN PERIOD [2/6]

$y[n]*y[-n]$ $=x[n]*x[-n]+w[n]*x[-n]+x[n]*w[-n]+w[n]*w[-n]$
becomes: $r_y[n]=r_x[n]+r_w[n]=r_x[n]+\sigma^2\delta[n]$
since: $w[n]*x[-n]=x[n]*w[-n]=0$ by assumption (usually true).

Now: $r_x[N]=\sum_i x[i]x[i+N]=\sum x[i]^2=r_x[0]$
since: $x[n]$ is periodic with period=N.
Also: $r_x[0]=r_x[N]=r_x[2N]=\ldots >> r_x[n]$ for other n.

ESTIMATION OF UNKNOWN PERIOD [3/6]

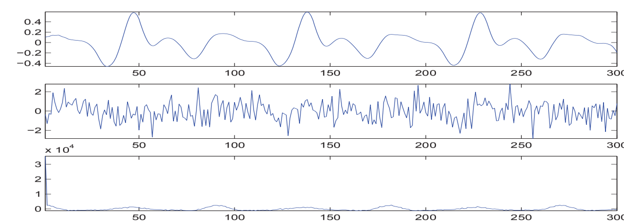
So: TO ESTIMATE N FROM $y[n]$:
1. Compute $r_y[n]$ from given $y[n]$ using FFT.
2a. $r_y[0]=\sigma^2+r_x[0]$ is larger than:
2b. $r_y[N]=r_y[2N]=r_y[3N]=\ldots=r_x[0]$
2c. $r_y[n] << r_y[N]=r_y[2N]=r_y[3N]=\ldots$ for other n.
3a. Look for 1^{st} peak in $r_y[n]$ with height$\approx r_y[0]-\sigma^2$.
3b. Location of this 1^{st} peak is period N of $x[n]$.

ESTIMATION OF UNKNOWN PERIOD [4/6]

EX: Noise added to signal of trumpet playing unknown note.
Next: Plots: noiseless trumpet; noisy trumpet; autocorrelation.

Get: $r_y[n]$ has period=90 (not 45–look at it carefully).
So: Frequency=$44100\frac{\text{SAMPLE}}{\text{SECOND}}/90=490$ Hz=note "B."

ESTIMATION OF UNKNOWN PERIOD [5/6]



ESTIMATION OF UNKNOWN PERIOD [6/6]

Given: Trumpet signal in `X` has length=32768
Add noise: `load trumpet;Y=X+randn(1,32768);`
$r_y[n]$: `R=real(ifft(abs(fft(Y,65536)).^2));`
Noiseless: `subplot(411),plot(X(1:300)),axis tight`
Noisy: `subplot(412),plot(Y(1:300)),axis tight`
Autocorre: `subplot(413),plot(R(1:300)),axis tight`

CROSS-CORRELATION [1/2]

DEF: Cross-correlation $r_{xy}[n]$ of $x[n]$ and $y[n]$:
$r_{xy}[n]=$ $x[n]*y[-n]=\sum_{i=-\infty}^{\infty} x[i]y[i-n]$. Also,
$r_{yx}[n]=$ $y[n]*x[-n]=\sum_{i=-\infty}^{\infty} x[i]y[i+n]=r_{xy}[-n]$.

DTFT: $R_{xy}(e^{j\omega})=X(e^{j\omega})Y(e^{-j\omega})=X(e^{j\omega})Y(e^{j\omega})^*$.
Given: $x[n],y[n]$ stored in row vectors `X,Y`. Compute using Matlab:
Then: `L=length([X Y]);N=nextpow2(L-1);`
and: `R=real(ifft(fft(X,N).*conj(fft(Y,N))));`

CROSS-CORRELATION [2/2]

EXAMPLE: COMPUTING $r_{xy}[n]$:

Given: $x[n]=\{3,1,4\}$ and $y[n]=\{2,7,1\}$.
$r_{xy}[n]=$ $\{3\cdot 1, 3\cdot 7+1\cdot 1, 3\cdot 2+1\cdot 7+4\cdot 1, 4\cdot 7+1\cdot 2, 4\cdot 2\}$
$r_{xy}[n]=$ $\{3,1,4\}*\{1,7,2\}=\{3,22,\underline{17},30,8\}$.
Note: $r_{xy}[0]=\sum_i x[i]y[i]=$X·Y (dot product).
Note: Autocorrelation=cross-correlation with itself.

TIME DELAY ESTIMATION [1/3]

Given: Observe $y[n]=x[n\text{–}D]+w[n]$
where: $x[n]$ is delayed by an unknown delay=D.
and: $w[n]$=0-mean white Gaussian noise with strength σ^2.
Goal: Estimate D from noisy observations $y[n]$.

Why? $x[n]$ is a known radar or sonar pulse.
Goal: How far away is airplane or whale or sub?
Then: Distance=$Dc/2$ where c=wave speed.

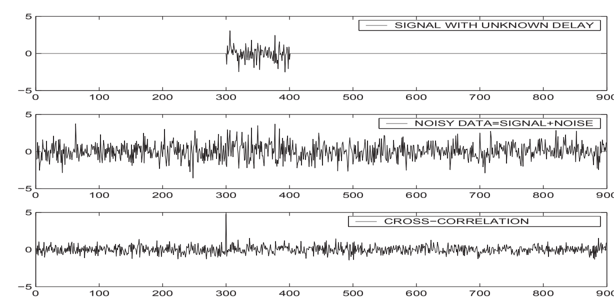
TIME DELAY ESTIMATION [2/3]

$r_{yx}[n]=$ $y[n]*x[-n]=x[n\text{–}D]*x[-n]+w[n]*x[-n]=r_x[n\text{–}D]$
since: $x[n]*x[-n]=r_x[n]$ and $w[n]*x[-n]=0$ by assumption
and: Delaying either of 2 signals delays their convolution.

Now: $r_x[n]$ is even, and usually $r_x[0]>>r_x[n]$ for $n\neq 0$.
So: $r_{yx}[n]$ has a peak at n=D=unknown delay.

1. Compute $r_{yx}[n]=y[n]*x[-n]=\sum_{i=-\infty}^{\infty} x[i]y[i+n]$
2. Look for peak. Location=unknown delay D.

TIME DELAY ESTIMATION [3/3]



SIGNAL SELECTION

Given: $y[n]=\begin{cases} x_1[n-D_1]+\text{noise} & \text{OR} \\ x_2[n-D_2]+\text{noise} & \text{OR}\ldots \end{cases}$
Goal: Determine which one of the $x_i[n]$ is present.

Idea: Cross-correlate $y[n]$ with each $x_i[n]$.
Then: One of the $r_{yx_i}[n]$ will be large at n=D_i.
So: That $x_i[n]$ is the signal that is present.
Note: Works best when $x_i[n]$ are orthogonal (next slide).

SIGNAL SIMILARITY

DEF: $\rho=(\sum_i x[i]y[i])/\sqrt{\sum_i x[i]^2\sum_i y[i]^2}=$ CORRELATION COEFFICIENT.

Then: $\rho=1 \Leftrightarrow y[n]=ax[n], a>0$ and $\rho=-1 \Leftrightarrow y[n]=-ax[n]$.

EX: $x[n]=A\cos(\frac{2\pi}{N}n+\theta)$ and $y[n]=B\cos(\frac{2\pi}{N}n+\phi)$.
Then: $\rho=\cos(\theta\text{–}\phi)$ after much algebra.
EX: 180^o phase difference$\Leftrightarrow$ ρ=–1$\Leftrightarrow$multiply by $-\frac{B}{A}$.
And: 90^o phase difference$\Leftrightarrow$ ρ=0$\Leftrightarrow$orthogonal (sin and cos).