TOPICS FOR TODAY'S LECTURE

1. Computing Convolution:
   a. (Finite)*(Finite) lengths
   b. (Finite)*(Infinite) lengths
2. BIBO stability condition
   a. Proof of IFF theorem
3. ARMA difference equations

FINITE*FINITE LENGTHS: EX [1/6]

Goal: Compute $\{\underline{1}, 2, 3\} * \{\underline{4}, 5, 6, 7\}$.

Note: Length y[n]=3+4–1=6.

Why? Input $x[n] = \{\underline{4}, 5, 6, 7\} \rightarrow \boxed{\mathbf{h[n]}} \rightarrow y[n]$
where: Impulse response $h[n] = \{\underline{1}, 2, 3\}$
That is: System is y[n]=x[n]+2x[n–1]+3x[n–2].

FINITE*FINITE LENGTHS: EX [2/6]

Goal: Compute $\{\underline{1}, 2, 3\} * \{\underline{4}, 5, 6, 7\}$.

Since: Both h[n] and x[n] are causal
Use: $y[n]=\sum_{i=0}^{n} h[n-i]x[i]$. Endpoints:

n=0: $y[0]=\sum_{i=0}^{0} h[0-i]x[i]=h[0]x[0]=(1)(4)=04$.

n=5: $y[5]=\sum_{i=3}^{3} h[5-i]x[i]=h[2]x[3]=(3)(7)=21$.

since: For i=0,1,2,4,5 have h[5–i]x[i]=0.

FINITE*FINITE LENGTHS: EX [3/6]

Goal: Compute $\{\underline{1}, 2, 3\} * \{\underline{4}, 5, 6, 7\}$.

n=0: $y[0]=\sum_{i=0}^{0} h[0-i]x[i]=h[0]x[0]=(1)(4)=04$.

n=1: $y[1]=\sum_{i=0}^{1} h[1-i]x[i]=h[1]x[0]+h[0]x[1]=(2)(4)+(1)(5)=13$.

n=4: $y[4]=\sum_{i=2}^{3} h[4-i]x[i]=h[2]x[2]+h[1]x[3]=(3)(6)+(2)(7)=32$.

since: For i=0,1,4 have h[4–i]x[i]=0.

$y[5]=\sum_{i=3}^{3} h[5-i]x[i]=h[2]x[3]=(3)(7)=21$.

FINITE*FINITE LENGTHS: EX [4/6]

Goal: Compute $\{\underline{1}, 2, 3\} * \{\underline{4}, 5, 6, 7\}$.

n=0: $y[0]=\sum_{i=0}^{0} h[0-i]x[i]=h[0]x[0]=(1)(4)=04$.

n=1: $y[1]=\sum_{i=0}^{1} h[1-i]x[i]=h[1]x[0]+h[0]x[1]=(2)(4)+(1)(5)=13$.

n=2: y[2]=h[2]x[0]+h[1]x[1]+h[0]x[2]=(3)(4)+(2)(5)+(1)(6)=28.

n=3: y[3]=h[2]x[1]+h[1]x[2]+h[0]x[3]=(3)(5)+(2)(6)+(1)(7)=34.

n=4: $y[4]=\sum_{i=2}^{3} h[4-i]x[i]=h[2]x[2]+h[1]x[3]=(3)(6)+(2)(7)=32$.

n=5: $y[5]=\sum_{i=3}^{3} h[5-i]x[i]=h[2]x[3]=(3)(7)=21$.

Answer: $\{\underline{4}, 13, 28, 34, 32, 21\}$.

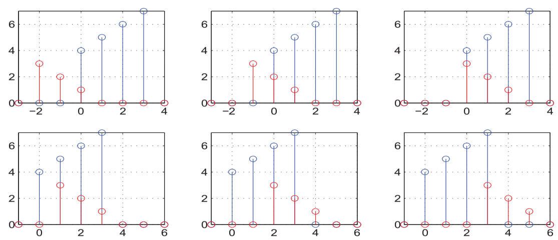
FINITE*FINITE LENGTHS: EX [5/6]
OTHER METHODS FOR CONVOLUTION:

1. Multiply polynomials:
   $(1z^2+2z+3)(4z^3+5z^2+6z+7)=$
   $4z^5+13z^4+28z^3+34z^2+32z+21$
   This is z-transform (except uses z^{-1}).
2. Flip method (see next slide)

Check: (1+2+3)(4+5+6+7)=(6)(22)=132.
Equals: $\sum y[n]=4+13+28+34+32+21=132$.

FINITE*FINITE LENGTHS: EX [6/6]

Sum: Product of h[n–i] (red) and x[n] (blue) in each plot.
Where: n=0,1,2: top row (left-to-right). n=3,4,5: bottom row.



FINITE*INFINITE LENGTHS: EX #1 [1/4]

Goal: Compute $\{\underline{2},-1,3\} * (\frac{1}{2})^n u[n] = ?$

Why? Input $x[n] = \{\underline{2},-1,3\} \rightarrow \boxed{\mathbf{h[n]}} \rightarrow y[n]$
where: Impulse response $h[n] = (\frac{1}{2})^n u[n]$.
That is: System is $y[n] - \frac{1}{2}y[n-1] = x[n]$.
Since: Will show this later in lecture.
OR: Exchange input and impulse response.

FINITE*INFINITE LENGTHS: EX #1 [4/4]

Goal: $\{\underline{2},-1,3\} * (\frac{1}{2})^n u[n] =$

Rewrite: $(2\delta[n] - 1\delta[n-1] + 3\delta[n-2]) * (\frac{1}{2})^n u[n] =$

Distrib.: $(\frac{1}{2})^{n-1}u[n] - 1(\frac{1}{2})^{n-1}u[n-1] + 3(\frac{1}{2})^{n-2}u[n-2] =$

Answer: $2\delta[n] + 3(\frac{1}{2})^{n-2}u[n-2]$ (1^{st} two terms sum to $2\delta[n]$).

Answer: $\{\underline{2}, 0, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots\}$.

FINITE*INFINITE LENGTHS: EX #2 [2/2]

Goal: $\{\underline{1},-3,2\} * u[n] =$

Rewrite: $(\delta[n] - 3\delta[n-1] + 2\delta[n-2]) * u[n] =$

Distrib.: $u[n] - 3u[n-1] + 2u[n-2] =$

Simplify: $= \begin{cases} 1+0+0=1 & \text{for } n=0 \\ 1-3+0=-2 & \text{for } n=1 \\ 1-3+2=0 & \text{for } n\geq 2 \end{cases} = \{\underline{1},-2\}.$

BIBO STABILITY CONDITION

THM. An LTI system is BIBO stable
IFF: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
That is: $h[n]$ is *absolutely summable.*

IFF: If and Only If: same as $\Leftrightarrow$

Corollary: $y[n]=b_0x[n]+b_1x[n-1]+\ldots+b_Mx[n-M]$ (MA)
is **always** BIBO stable for **any** $\{b_0 \ldots b_M\}$

since: $h[n] = \{\underline{b_0}, b_1, b_2 \ldots b_M\}$ (just read off)
and: $|b_0| + \ldots + |b_M|$ is finite (since M is finite).

BIBO STABILITY CONDITION: EXS [1/3]

EX #1: $h[n] = \{2, 3, -4\}$. Is the LTI system BIBO stable?
$\rightarrow \sum |h[n]| = |2|+|3|+|-4| = 9 < \infty \rightarrow \substack{\text{BIBO} \\ \text{stable}}$.

General: $y[n]=b_0x[n]+b_1x[n-1]+\ldots+b_Mx[n-M]$
Implies: $h[n] = \{\underline{b_0}, b_1 \ldots b_M\}$ (FIR) always BIBO stable.
That is: Finite Impulse Response is always BIBO stable.

BIBO STABILITY CONDITION: EXS [2/3]

EX #2: $h[n] = (-\frac{1}{2})^n u[n]$
$\rightarrow \sum |h[n]| = \sum |-\frac{1}{2}|^n = \frac{1}{1-0.5} < \infty \rightarrow \frac{\text{BIBO}}{\text{stable}}$.

General: $h[n] = Cp^n u[n]$ is BIBO stable if $|p| < 1$.
Also: $h[n] = \sum_{i=1}^{M} C_i p_i^n u[n]$ is BIBO stable
if: $|p_i| < 1$ for all $i = 1 \ldots M$ (M finite).

since: $\sum |h[n]| = \sum |\sum_{i=1}^{M} C_i p_i^n u[n]| \leq \sum_{i=1}^{M} \frac{|C_i|}{1-|p_i|}$.
That is: Sum of *decaying* geometrics is BIBO stable.

BIBO STABILITY CONDITION: EXS [3/3]

EX #3: $h[n] = \{\underline{1}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5} \ldots\}$
$\sum h[n] = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \log_e(2)$
$\sum |h[n]| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \rightarrow \infty$
So: $h[n]$ is summable but not *absolutely* summable!
Hence: LTI system with this $h[n]$ is NOT BIBO stable.

BIBO STABILITY CONDITION: PROOF [1/5]

PROOF: Since IFF, proof has 2 parts.
$\Leftarrow$: **Suppose:** $\sum_{n=-\infty}^{\infty} |h[n]| = L$.
Goal: *Prove* BIBO stability. How to do that?

BIBO STABILITY CONDITION: PROOF [2/5]

$\Leftarrow$: **Suppose:** $\sum_{n=-\infty}^{\infty} |h[n]| = L$.
Goal: *Prove* BIBO stability:
Suppose: *Given* **BI:** $|x[n]| \leq M$.
Goal: *Prove* **BO:** $|y[n]|$ bounded.

BIBO STABILITY CONDITION: PROOF [3/5]

Suppose: $\sum_{n=-\infty}^{\infty} |h[n]| = L$ and $|x[n]| \leq M$.
Goal: Prove $|y[n]|$ is bounded.

Then: $|y[n]| = |\sum_{i=-\infty}^{\infty} h[i]x[n-i]|$
$\leq \sum_{i=-\infty}^{\infty} |h[i]| \cdot |x[n-i]|$ (triangle inequality)
$\leq \sum_{i=-\infty}^{\infty} |h[i]| M = LM$ Q.E.D.

BIBO STABILITY CONDITION: PROOF [4/5]

PROOF: Since IFF, proof has 2 parts.
$\Rightarrow$: **Suppose:** $\sum_{n=-\infty}^{\infty} |h[n]| \rightarrow \infty$
Goal: *Prove* NOT BIBO stable. How to do that?
Soln: Find a counterexample.

BIBO STABILITY CONDITION: PROOF [5/5]

⇒: Suppose: $\sum_{n=-\infty}^{\infty}|h[n]| \rightarrow \infty$
Goal: *Prove* NOT BIBO stable. Find a counterexample.
Soln: Let $x[n] = \text{sign}\{h[-n]\} = \pm 1$.

Then: $y[0] = \sum_{i=-\infty}^{\infty} h[i]x[0-i] = \sum_{i=-\infty}^{\infty} h[i]x[-i]$

$= \sum_{i=-\infty}^{\infty} h[i]\text{sign}\{h[i]\} = \sum_{i=-\infty}^{\infty}|h[i]| \rightarrow \infty$.

So: Bounded input $x[n] = \text{sign}\{h[-n]\} = \pm 1$
yields: Unbounded output $y[0] = \sum_{i=-\infty}^{\infty}|h[i]| \rightarrow \infty$. QED.

ARMA DIFFERENCE EQUATIONS: MA PART

MA: y[n]=b_0x[n]+b_1x[n–1]+…+b_qx[n–q] (Moving Average)
Huh? Present output=weighted average of q *most recent* inputs.
Note: Equivalent to y[n]=b[n]*x[n] where $b[k] = b_k, 0 \leq k \leq q$.

FIR: Finite Impulse Response ⇔ $h[n]$ has finite *duration*.
Note: Any MA system is also an FIR system, and vice-versa.
EX: System is y[n]=x[n]+2x[n–1]+3x[n–2]⇔h[n]=$\{\underline{1}, 2, 3\}$.

ARMA DIFFERENCE EQUATIONS: AR PART

AR: y[n]+a_1y[n–1]+…+a_py[n–p]=x[n] (AutoRegression)
Huh? Present output=weighted sum of p *most recent* **outputs**.
Note: Compute $y[n]$ *recursively* from its p most recent values.

IIR: Infinite Impulse Response ⇔ $h[n]$ not finite duration.
But can implement using finite set of *coefficients* a_i.

EX: $h[n] = a^n u[n] = a^n$ for $n \geq 0$ and $|a| < 1$ is stable and IIR.
⇔ System is y[n]–ay[n–1]=x[n]. Will show this slide after next.

ARMA DIFFERENCE EQUATIONS

ARMA:
$$\underbrace{\sum_{i=0}^{p} a_i y[n-i]}_{\text{AUTOREGRESSIVE}} = \underbrace{\sum_{i=0}^{q} b_i x[n-i]}_{\text{MOVING AVERAGE}}$$

Analogous to *differential equation* in continuous time:
$\frac{d^p y}{dt^p}+a_1\frac{d^{p-1}y}{dt^{p-1}}+\ldots+a_p y(t)=b_0\frac{d^q x}{dt^q}+b_1\frac{d^{q-1}x}{dt^{q-1}}+\ldots+b_q x(t)$.
Note: Coefficients a_i and b_i are *not directly* analogous here.

RECURSIVE COMPUTATION OF h[n] [2/2]

Goal: Compute impulse response $h[n]$ of system y[n]–$\frac{1}{2}$y[n–1]=3x[n].
Soln: Compute recursively $h[n] - \frac{1}{2}h[n-1] = 3\delta[n] = 0$ if $n > 0$.

n=0: $h[0] - \frac{1}{2}h[-1] = 3\delta[0] \rightarrow h[0] - \frac{1}{2}(0) = 3(1) \rightarrow h[0] = 3$.

n=1: $h[1] - \frac{1}{2}h[0] = 3\delta[1] \rightarrow h[1] - \frac{1}{2}(3) = 3(0) \rightarrow h[1] = \frac{3}{2}$.

n=2: $h[2] - \frac{1}{2}h[1] = 3\delta[2] \rightarrow h[2] - \frac{1}{2}(\frac{3}{2}) = 3(0) \rightarrow h[2] = \frac{3}{4}$.

n=3: $h[3] - \frac{1}{2}h[2] = 3\delta[3] \rightarrow h[3] - \frac{1}{2}(\frac{3}{4}) = 3(0) \rightarrow h[3] = \frac{3}{8}$.

$h[n] = 3(\frac{1}{2})^n u[n] = 3(\frac{1}{2})^n$ for $n \geq 0$. *Geometric* signal.

RECURSIVE IMPLEMENTATION OF ARMA

ARMA: $\sum_{i=0}^{p} a_i y[n-i] = \sum_{i=0}^{q} b_i x[n-i]$
Implement recursively as:
y[n]=$\sum_{i=0}^{q} b_i x[n-i]$–$\sum_{i=1}^{p} a_i y[n-i]$

EX: y[n]+2y[n–1]+3y[n–2]=4x[n]+5x[n–1]
Implement recursively as:
y[n]=4x[n]+5x[n–1]–2y[n–1]–3y[n–2]

Need: 3 storage registers; 4 mult-and-adds.
Matlab: `>>Y=filter([4 5],[1 2 3],X);` `X`=input vector.